

On the Geodesic Centers of Polygonal Domains^{*}

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Abstract. In this paper, we study the problem of computing Euclidean geodesic centers of a polygonal domain \mathcal{P} with a total of n vertices. We discover many interesting observations. We give a necessary condition for a point being a geodesic center. We show that there is at most one geodesic center among all points of \mathcal{P} that have topologically-equivalent shortest path maps. This implies that the total number of geodesic centers is bounded by the combinatorial size of the shortest path map equivalence decomposition of \mathcal{P} , which is known to be $O(n^{10})$. One key observation is a π -range property on shortest path lengths when points are moving. With these observations, we propose an algorithm that computes all geodesic centers in $O(n^{11} \log n)$ time. Previously, an algorithm of $O(n^{12+\epsilon})$ time was known for this problem, for any $\epsilon > 0$.

1 Introduction

Let \mathcal{P} be a polygonal domain with a total of h holes and n vertices, i.e., \mathcal{P} is a multiply-connected region whose boundary is a union of n line segments, forming $h + 1$ closed polygonal cycles. A simple polygon is a special case of a polygonal domain with $h = 0$. For any two points s and t , a *shortest path* or *geodesic path* from s to t is a path in \mathcal{P} whose Euclidean length is minimum among all paths from s to t in \mathcal{P} ; we let $d(s, t)$ denote the Euclidean length of any shortest path from s to t and we also say that $d(s, t)$ is the *shortest path distance* or *geodesic distance* from s to t .

A point s is a *geodesic center* of \mathcal{P} if s minimizes the value $\max_{t \in \mathcal{P}} d(s, t)$, i.e., the maximum geodesic distance from s to all points of \mathcal{P} . In this paper, we study the problem of computing the geodesic centers of \mathcal{P} .

The problem in simple polygons has been well studied. It is known that for any point in a simple polygon, its farthest point must be a vertex of the polygon [20]. It has been shown that the geodesic center in a simple polygon is unique and has at least two farthest points [18]. Due to these helpful observations, efficient algorithms for finding geodesic centers in simple polygons have been developed. Asano and Toussaint [2] first gave an $O(n^4 \log n)$ time algorithm for the problem, and later Pollack, Sharir, and Rote [18] solved the problem in $O(n \log n)$ time. It had been an open problem whether the problem is solvable in linear time until recently Ahn et al. [1] presented a linear-time algorithm for it.

Finding a geodesic center in a polygonal domain \mathcal{P} is much more difficult. This is partially due to that a farthest point of a point in \mathcal{P} may be in the interior of \mathcal{P} [3]. Also, it is easy to construct an example where the geodesic center of \mathcal{P} is not unique (e.g., see Fig. 1). Bae, Korman, and Okamoto [4] gave the first-known algorithm that can compute a geodesic center in $O(n^{12+\epsilon})$ time for any $\epsilon > 0$. They first showed that for any point its farthest points must be vertices of its shortest path map in \mathcal{P} . Then, they considered the shortest path map equivalence decomposition (or SPM-equivalence decomposition) [7], denoted by \mathcal{D}_{spm} ; for each cell of \mathcal{D}_{spm} , they computed the upper envelope of $O(n)$ graphs in three-dimensional space, which takes $O(n^{2+\epsilon})$ time [10], to search a geodesic center in the cell. Since the size of \mathcal{D}_{spm} is $O(n^{10})$ [7], their algorithm runs in $O(n^{12+\epsilon})$ time.

A concept closely related to the geodesic center is the *geodesic diameter*, which is the maximum geodesic distance over all pairs of points in \mathcal{P} , i.e., $\max_{s, t \in \mathcal{P}} d(s, t)$. In simple polygons, due to the property that there always exists a pair of vertices of \mathcal{P} whose geodesic distance is equal to the geodesic

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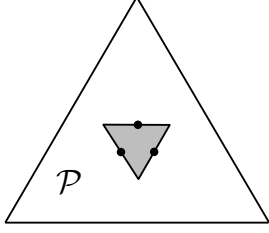


Fig. 1. The boundary of \mathcal{P} consists of an outer and an inner equilateral triangles with their geometric centers co-located. Each of the three thick points is a geodesic center of \mathcal{P} .

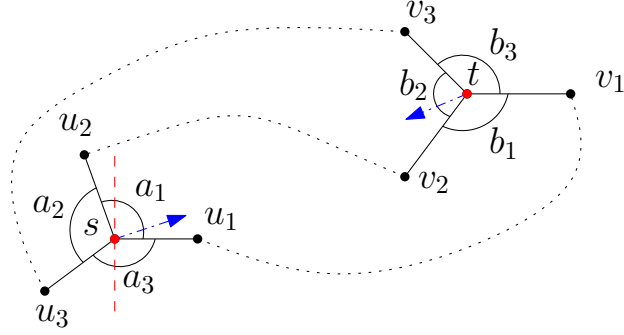


Fig. 2. Illustrating the π -range property. Suppose there are three shortest s - t paths through vertices u_i and v_i with $i = 1, 2, 3$, respectively. If s and t move along the blue arrows simultaneously (possibly with different “speeds”), then all three shortest paths strictly decreases (it is difficult to tell whether this is true from the figure, so these two blue directions here are only for illustration purpose). The special case happens when the six angles a_i and b_i for $i = 1, 2, 3$ satisfy $a_i = b_i$ for $i = 1, 2, 3$.

diameter, efficient algorithms have been given for computing the geodesic diameters. Chazelle [6] gave the first algorithm that runs in $O(n^2)$ time. Later, Suri [20] presented an $O(n \log n)$ -time algorithm. Finally, the problem was solved in $O(n)$ time by Hershberger and Suri [11]. Computing the geodesic diameter in a polygonal domain \mathcal{P} is much more difficult. It was shown in [3] that the geodesic diameter can be realized by two points in the interior of \mathcal{P} , in which case there are at least five distinct shortest paths between the two points. As for the geodesic center problem, this makes it difficult to discretize the search space. By an exhaustive-search method, Bae, Korman, and Okamoto [3] gave the first-known algorithm for computing the diameter of \mathcal{P} and the algorithm runs in $O(n^{7.73})$ or $O(n^7(\log n + h))$ time.

Refer to [5,8,13,15,16,17,19] for other variations of geodesic diameter and center problems (e.g., the L_1 metric and the link distance case).

1.1 Our Contributions

We conduct a “comprehensive” study on geodesic centers of \mathcal{P} . We discover many interesting observations, and some of them are even surprising. For example, we show that even if a geodesic center is in the interior of \mathcal{P} , it may have only one farthest point, which is somewhat counter-intuitive. We give a necessary condition for a point being a geodesic center. We also show that there is at most one geodesic center among all points of \mathcal{P} that have topologically-equivalent shortest path maps in \mathcal{P} . This immediately implies that the interior of each cell or each edge of the SPM-equivalence decomposition \mathcal{D}_{spm} can contain at most one geodesic center, and thus, the total number of geodesic centers of \mathcal{P} is bounded by the combinatorial size of \mathcal{D}_{spm} , which is known to be $O(n^{10})$ [7]. Previously, the only known upper bound on the total number of geodesic centers of \mathcal{P} is $O(n^{12+\epsilon})$, which is implied by the algorithm in [4].

These observations are all more or less due to an interesting observation, which we call the π -range property and is one key contribution of this paper. Here we demonstrate an application of the π -range property (see the paper for the details). Let s and t be two points in the interior of \mathcal{P} such that t is a farthest point of s in \mathcal{P} . Refer to Fig. 2 for an example. Suppose there are three shortest paths from s to t as shown in Fig. 2. The π -range property says that unless a special case happens, there exists an open range of exactly size π (e.g., delimited by the right open half-plane bounded by the vertical line through s in Fig. 2) such that if s moves along any direction in the range for an infinitesimal distance, we can always find a direction to move t such that the lengths of *all three* shortest paths strictly decrease. Further, if the special case does not happen, we can explicitly determine the above

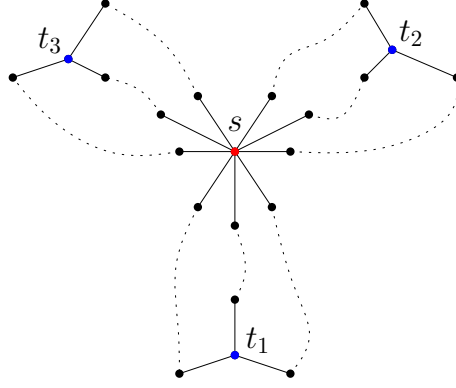


Fig. 3. Illustrating a geodesic center s with three farthest points t_1, t_2, t_3 such that all these four points are in the interior of \mathcal{P} . There are three shortest paths from s to each of t_1, t_2, t_3 .

range of size π . In fact, it is the special case that makes it possible for a geodesic center having only one farthest point.

With these observations, we propose an exhaustive-search algorithm to compute a set S of *candidate points* such that all geodesic centers must be in S . For example, refer to Fig. 3 for a schematic diagram, where a geodesic center s has three farthest points t_1, t_2, t_3 and all these four points are in the interior of \mathcal{P} . The nine shortest paths from s to t_1, t_2, t_3 provide a system of eight equations, which give eight (independent) constraints that can determine the four points s, t_1, t_2, t_3 if we consider the coordinates of these points as eight variables. This suggests our exhaustive-search approach to compute candidate points for such a geodesic center s (similar exhaustive-search approaches are also used before, e.g., [3,7]). However, if a geodesic center s has only one farthest point (e.g., Fig. 2), then we have only three shortest paths, which give only two constraints. In order to determine s and t , which have four variables, we need two more constraints. It turns out the π -range property (i.e., the special case) provides exactly two more constraints (on the angles as shown in Fig. 2). In this way, we can still compute candidate points for such s . Also, if a geodesic center has two farthest points, we will need one more constraint, which is also provided by the π -range property (the non-special case). Note that the previous exhaustive-search approaches [3,7] do not need the π -range property.

The number of candidate points in S is $O(n^{11})$. To find all geodesic centers from S , a straightforward solution is to compute the shortest path map for every point of S , which takes $O(n^{12} \log n)$ time in total. Again, with the help of the π -range property, we propose a *pruning algorithm* to eliminate most points from S in $O(n^{11} \log n)$ time such that none of the eliminated points is a geodesic center and the number of the remaining points of S is only $O(n^{10})$. Consequently, we can find all geodesic centers in additional $O(n^{11} \log n)$ time by computing their shortest path maps.

Although we improve the previous $O(n^{12+\epsilon})$ time algorithm in [4] by a factor of roughly $n^{1+\epsilon}$, the running time is still huge. We feel that our observations (in particular, the π -range property) may be more interesting than the algorithm itself. We suspect that these observations may also find applications in other related problems. The paper is lengthy and some discussions are quite tedious, which is mainly due to a considerable number of cases depending on whether a geodesic center and its farthest points are in the interior, on an edge, or at vertices of \mathcal{P} , although the essential idea is quite similar for all these cases.

The rest of the paper is organized as follows. In Section 2, we introduce notation and review some concepts. In Section 3, we give our observations. In particular, we prove the π -range property in Section 4. Our algorithm for computing the candidate points is presented in Section 5. Finally, we find all geodesic centers from the candidate points in Section 6.

2 Preliminaries

Consider any point $s \in \mathcal{P}$. Let $d_{\max}(s)$ denote the maximum geodesic distance from s to all points of \mathcal{P} , i.e., $d_{\max}(s) = \max_{t \in \mathcal{P}} d(s, t)$. A point $t \in \mathcal{P}$ is a *farthest point* of s if $d(s, t) = d_{\max}(s)$. We use $F(s)$ to denote the set of all farthest points of s in \mathcal{P} . For any two points p and q in \mathcal{P} , for convenience of discussions, we say that p is *visible* to q if the line segment \overline{pq} is in \mathcal{P} and the interior of \overline{pq} does not contain any vertex of \mathcal{P} . We use $|pq|$ to denote the (Euclidean) length of any line segment \overline{pq} . Note that two points s and t in \mathcal{P} may have more than one shortest path between them, and if not specified, we use $\pi(s, t)$ to denote any such shortest path.

For simplicity of discussion, we make a general position assumption that any two vertices of \mathcal{P} have only one shortest path and no three vertices of \mathcal{P} are on the same line.

Denote by \mathcal{I} the set of all interior points of \mathcal{P} , \mathcal{V} the set of all vertices of \mathcal{P} , and E the set of all relatively interior points on the edges of \mathcal{P} (i.e., E is the boundary of \mathcal{P} minus \mathcal{V}).

Shortest path maps. Given a point $s \in \mathcal{P}$, a *shortest path map* of s [7], denoted by $SPM(s)$, is a decomposition of \mathcal{P} into regions (or cells) such that in each cell σ , the combinatorial structures of shortest paths from s to all points t in σ are the same, and more specifically, the sequence of obstacle vertices along $\pi(s, t)$ is fixed for all t in σ . Further, the *root* of σ , denoted by $r(\sigma)$, is the last vertex of $\mathcal{V} \cup \{s\}$ in the path $\pi(s, t)$ for any point $t \in \sigma$ (hence $\pi(s, t) = \pi(s, r(\sigma)) \cup r(\sigma)t$; note that $r(\sigma)$ is s if s is visible to t). As in [7], we classify each edge of σ into three types: a portion of an edge of \mathcal{P} , an *extension segment*, which is a line segment extended from $r(\sigma)$ along the opposite direction from $r(\sigma)$ to the vertex of $\pi(s, t)$ preceding $r(\sigma)$, and a *bisector curve/edge* that is a hyperbolic arc. For each point t in a bisector edge of $SPM(s)$, t is on the common boundary of two cells and there are two shortest paths from s to t through the roots of the two cells, respectively (and neither path contains both roots). The *vertices* of $SPM(s)$ include $\mathcal{V} \cup \{s\}$ and all intersections of edges of $SPM(s)$. If a vertex t of $SPM(s)$ is an intersection of two or more bisector edges, then there are more than two shortest paths from s to t . The map $SPM(s)$ has $O(n)$ vertices, edges, and cells, and can be computed in $O(n \log n)$ time [12]. It was shown [4] that any farthest point of s in \mathcal{P} must be a vertex of $SPM(s)$.

For differentiation, we will refer to the vertices of \mathcal{V} as *polygon vertices* and refer to the edges of E as *polygon edges*.

The SPM-equivalence decomposition \mathcal{D}_{spm} of \mathcal{P} [7] is a subdivision of \mathcal{P} into regions such that for all points s in the interior of the same region or edge of \mathcal{D}_{spm} , the shortest path maps $SPM(s)$ are topologically equivalent. Chiang and Mitchell [7] showed that the combinatorial complexity of \mathcal{D}_{spm} is bounded by $O(n^{10})$ and \mathcal{D}_{spm} can be computed in $O(n^{11})$ time.

Directions and ranges. In this paper, we will have intensive discussions on moving points along certain directions. For any direction r , we represent r by the angle $\alpha(r) \in [0, 2\pi)$ counterclockwise from the positive direction of the x -axis. For convenience, whenever we are talking about an angle α , unless otherwise specified, depending on the context we may refer to any angle $\alpha + 2\pi \cdot k$ for $k \in \mathbb{Z}$. For any two angles α_1 and α_2 with $\alpha_1 \leq \alpha_2 < \alpha_1 + 2\pi$, the interval $[\alpha_1, \alpha_2]$ represents a *direction range* that includes all directions whose angles are in $[\alpha_1, \alpha_2]$, and $\alpha_2 - \alpha_1$ is called the *size* of the range. Note that the range can be open (e.g., (α_1, α_2)) and the size of any direction range is no more than 2π .

Consider a half-plane h whose bounding line is through a point s in the plane. We say h *delimits* a range of size π of directions for s that consists of all directions along which s will move towards inside h . If h is an open half-plane, then the range is open as well.

A direction r for $s \in \mathcal{P}$ is called a *free direction* of s if we move s along r for an infinitesimal distance then s is still in \mathcal{P} . We use $R_f(s)$ to denote the range of all free directions of s . Clearly, if $s \in \mathcal{I}$, $R_f(s)$ contains all directions; if $s \in E$, $R_f(s)$ is a (closed) range of size π ; if $s \in V$, $R_f(s)$ is delimited by the two incident polygon edges of s .

3 Observations

Consider any point $s \in \mathcal{P}$ and let t be any farthest point of s . Recall that t is a vertex of $SPM(s)$ [4]. Suppose we move s infinitesimally along a free direction r to a new point s' . Since $|ss'|$ is infinitesimal, we can assume that s and s' are in the same cell σ of \mathcal{D}_{spm} . Further, if s is in the interior of σ , then s' is also in the interior of σ .

Regardless of whether s is in the interior of σ or not, there is a vertex $t' \in SPM(s')$ corresponding to the vertex t of $SPM(s)$ in the following sense [7]: If the line segment $\overline{s't'}$ is a shortest path from s' to t' , then \overline{st} is a shortest path from s to t ; otherwise, if $s', u_1, u_2, \dots, u_k, t'$ is the sequence of the vertices of $\mathcal{V} \cup \{s', t'\}$ in a shortest path from s' to t' , then $s, u_1, u_2, \dots, u_k, t$ is also the sequence of the vertices of $\mathcal{V} \cup \{s, t\}$ in a shortest path from s to t .

In the case that s is on the boundary of σ while s' is in the interior of σ , there might be more than one such vertex $t' \in \mathcal{D}_{spm}$ corresponding to t (refer to [7] for the details) and we use $M_t(s')$ to denote the set of all such vertices t' . We should point out that although a vertex in $SPM(s)$ may correspond to more than one vertex in $SPM(s')$, any vertex in $SPM(s')$ can correspond to one and only one vertex in $SPM(s)$ (because s' is always in the interior of σ).

We introduce the following definition which is crucial to the paper.

Definition 1. A free direction r is called an *admissible direction* of s with respect to t if as we move s infinitesimally along r to a new point s' , $d(s', t') < d(s, t)$ holds for each $t' \in M_t(s')$.

For any $t \in F(s)$, let $R(s, t)$ denote the set of all admissible directions of s with respect to t ; let $R(s) = \bigcap_{t \in F(s)} R(s, t)$. The following Lemma 1, which gives a necessary condition for a point being a geodesic center of \mathcal{P} , explains why we consider admissible directions.

Before proving Lemma 1, we introduce some notation and Observation 1.

Consider any two points s and t in \mathcal{P} . Suppose the vertices of $\mathcal{V} \cup \{s, t\}$ along a shortest s - t path $\pi(s, t)$ are $s = u_0, u_1, \dots, u_k = t$. According to our definition on the “visibility”, s is visible to t if and only if $k = 1$. If s is not visible to t , then $k \neq 1$ and we call u_1 an *s-pivot* and u_{k-1} a *t-pivot* of $\pi(s, t)$. It is possible that there are multiple shortest paths between s and t , and thus there might be multiple *s*-pivots and *t*-pivots for (s, t) . We use $U_s(t)$ and $U_t(s)$ to denote the sets of all *s*-pivots and *t*-pivots for (s, t) , respectively. Note that according to our above definition, for any $u \in U_s(t)$, the line segment \overline{su} does not contain any polygon vertex in its interior.

We have the following observation. Similar results have been given in [3].

Observation 1 Suppose t is a farthest point of a point s .

1. If t is in \mathcal{I} , then $|U_t(s)| \geq 3$ and t must be in the interior of the convex hull of the vertices of $U_t(s)$.
2. If t is in E , say, $t \in e$ for a polygon edge e of E , then $|U_t(s)| \geq 2$ and $U_t(s)$ has at least one vertex in the open half-plane bounded by the supporting line of e and containing the interior of \mathcal{P} in the small neighborhood of e . Further, $U_t(s)$ has at least one vertex in each of the two open half-planes bounded by the line through t and perpendicular to e .

Proof: The proof is similar to those in [3]. The main idea is that if the conditions are not satisfied, then we can always find a point t' further to s than t , incurring contradiction. \square

Lemma 1. If s is a geodesic center of \mathcal{P} , then $R(s) = \emptyset$.

Proof: Assume to the contrary that $R(s) \neq \emptyset$. Let r be any direction in $R(s)$. Then, r is in $R(s, t)$ for each $t \in F(s)$. In other words, r is an admissible direction of s with respect to each $t \in F(s)$. Suppose we move s infinitesimally along r to a new point s' . In the following, we show that $d_{\max}(s') < d_{\max}(s)$, which contradicts with that s is a geodesic center.

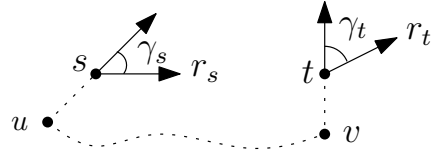
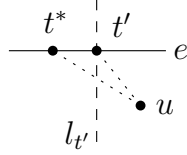


Fig. 4. Illustrating the proof of Lemma 1 for the case $t' \in E$: **Fig. 5.** Illustrating the definitions of the angles γ_s and γ_t . $|ut'| < |ut^*|$.

Consider any $t' \in F(s')$, i.e., t' is a farthest point of s' . To prove $d_{\max}(s') < d_{\max}(s)$, it is sufficient to show that $d(s', t') < d_{\max}(s)$.

If t' is in $M_t(s')$ for any $t \in F(s)$, then since r is an admissible direction of s with respect to t , it holds that $d(s', t') < d(s, t) = d_{\max}(s)$. In the following, we assume t' is not in $M_t(s')$ for any $t \in F(s)$.

Let $V(s)$ be the set of all vertices of $SPM(s)$ that are not in $F(s)$. Let $\delta = d_{\max}(s) - \max_{t \in V(s)} d(s, t)$. Note that since $|V(s)|$ is finite, the value δ is well-defined. Also note that the value δ is fixed and does not depend on s' . Since s moves infinitesimally to s' , we can assume $|ss'| < \delta$.

Since t' is a farthest point of s' , t' must be a vertex of $SPM(s')$. Let t^* be the vertex of $SPM(s)$ corresponding to t' . Note that although a vertex of $SPM(s)$ may correspond to multiple vertices of $SPM(s')$, a vertex of $SPM(s')$ corresponds to one and only one vertex of $SPM(s)$. Hence, the vertex t^* is unique, and further, $U_{t'}(s') \subseteq U_{t^*}(s)$. Since t' is not in $M_t(s')$ for any $t \in F(s)$, we know that t^* is not in $F(s)$ but in $V(s)$. Therefore, $d(s, t^*) \leq \max_{t \in V(s)} d(s, t) = d_{\max}(s) - \delta$.

Since $\overline{ss'} \in \mathcal{P}$, it holds that $d(s', t') \leq |s's| + d(s, t')$. Therefore, if we can prove $d(s, t') \leq d(s, t^*)$, since $|ss'| < \delta$ and $d(s, t^*) \leq d_{\max}(s) - \delta$, we can obtain $d(s', t') \leq |s's| + d(s, t') < \delta + d_{\max}(s) - \delta = d_{\max}(s)$. In the sequel, we prove $d(s, t') \leq d(s, t^*)$.

If $t' \in \mathcal{V}$, then t^* must be t' [7]. Hence, $d(s, t') = d(s, t^*)$ and thus $d(s, t') \leq d(s, t^*)$ trivially follows. In the following, we assume $t' \notin \mathcal{V}$. Thus, t' is either in E or \mathcal{I} .

Recall that $U_{t'}(s') \subseteq U_{t^*}(s)$. To prove $d(s, t') \leq d(s, t^*)$, it is sufficient to find a vertex $u \in U_{t'}(s')$ such that $|ut'| \leq |ut^*|$ because $d(s, t') = d(s, u) + |ut'|$ and $d(s, t^*) = d(s, u) + |ut^*|$. To this end, we will make use of Observation 1.

1. If $t' \in E$, let e be the polygon edge of E that contains t' . Since t' is a farthest point of s' , by Observation 1, there must be a vertex of $U_{t'}(s')$ on either open half-plane bounded by $l_{t'}$, where $l_{t'}$ is the line through t' and perpendicular to e . Since $t' \in E$, t^* is either on e or an endpoint of e [7]. In either case, one open half-plane bounded by $l_{t'}$ contains t^* and the other does not (e.g., see Fig. 4). Let u be the vertex of $U_{t'}(s')$ in the open half-plane bounded by $l_{t'}$ that does not contain t^* . Clearly, $|ut'| \leq |ut^*|$ holds.
2. If $t' \in \mathcal{I}$, then since t' is a farthest point of s' , by Observation 1, t' must be in the interior of the convex hull of all vertices of $U_{t'}(s')$. Then, regardless of whatever position of t^* is, there must be a vertex $u \in U_{t'}(s')$ on the convex hull such that $|ut'| \leq |ut^*|$ holds.

The lemma is thus proved. \square

As explained in Section 1, we will compute candidate points for geodesic centers. As a necessary condition, Lemma 1 will be helpful for computing those candidate points.

Consider any point $s \in \mathcal{P}$. Let t be a farthest point of s . We need to find a way to determine the admissible range $R(s, t)$. To this end, we will give a sufficient condition for a direction being in $R(s, t)$. We first assume that s is not visible to t , and as will be seen later, the other case is trivial.

Let u and v respectively be the s -pivot and the t -pivot of (s, t) in a shortest s - t path $\pi(s, t)$. Clearly, $d(s, t) = |su| + d(u, v) + |vt|$. We define $d_{u,v}(s, t) = |su| + d(u, v) + |vt|$ as a function of $s \in \mathbb{R}^2$ and $t \in \mathbb{R}^2$. Suppose we move s along a free direction r_s with the unit speed and move t along a free direction r_t with a speed $\tau \geq 0$. Let γ_s denote the smaller angle between the following two rays originated from s (e.g., see Fig. 5): one with direction r_s and one with direction from u to s . Similarly,

let γ_t denote the smaller angle between the following two rays originated from t : one with direction r_t and one with direction from v to t . In fact, as discussed in [3], if we consider $d(s, t)$ as a four-variate function, the triple (r_s, r_t, τ) corresponds to a vector ρ in \mathbb{R}^4 , and the directional derivative of $d_{u,v}(s, t)$ at $(s, t) \in \mathbb{R}^4$ along ρ , denoted by $d'_{u,v}(s, t)$, and the second directional derivative of $d_{u,v}(s, t)$ at (s, t) along ρ , denoted by $d''_{u,v}(s, t)$, are

$$d'_{u,v}(s, t) = \cos \gamma_s + \tau \cos \gamma_t, \quad d''_{u,v}(s, t) = \frac{\sin^2 \gamma_s}{|su|} + \tau \cdot \frac{\sin^2 \gamma_t}{|tv|}. \quad (1)$$

Since $\tau \geq 0$, $d''_{u,v}(s, t) \geq 0$ always holds. Further, if $\tau \neq 0$, then $d''_{u,v}(s, t) = 0$ if and only if $\sin^2 \gamma_s = \sin^2 \gamma_t = 0$, i.e., each of γ_s and γ_t is either 0 or π . In the following, in order to make the discussions more intuitive, we choose to use the parameters r_s , r_t , and τ instead of the vectors of \mathbb{R}^4 .

For each vertex $u \in U_s(t)$, there must be a vertex $v \in U_t(s)$ such that the concatenation of \overline{su} , $\pi(u, v)$, and \overline{vt} is a shortest path from s to t , and we call such a vertex v a *coupled t -pivot* of u (if u has more than one such vertex, then all of them are coupled t -pivots of u). Similarly, for each vertex $v \in U_t(s)$, we also define its coupled s -pivots in $U_s(t)$.

The following lemma provides a sufficient condition for a direction being an admissible direction for s with respect to t .

Lemma 2. *Suppose t is a farthest point of s and s is not visible to t .*

1. *For $t \in \mathcal{I}$, a free direction r_s is in $R(s, t)$ if there is a free direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with the unit speed and move t along r_t with speed τ , each vertex $v \in U_t(s)$ has a coupled s -pivot u with either $d'_{u,v}(s, t) < 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, t) = 0$.*
2. *For $t \in E$, a free direction r_s is in $R(s, t)$ if there is a free direction r_t for t that is parallel to the polygon edge of E containing t with a speed $\tau \geq 0$ such that when we move s along r_s with the unit speed and move t along r_t with speed τ , each vertex $v \in U_t(s)$ has a coupled s -pivot u with either $d'_{u,v}(s, t) < 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, t) = 0$.*
3. *For $t \in \mathcal{V}$, a free direction r_s is in $R(s, t)$ if we move s along r_s with the unit speed, each vertex $v \in U_t(s)$ has a coupled s -pivot u with either $d'_{u,v}(s, t) < 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, t) = 0$.*

Proof: Suppose we move s infinitesimally along r_s to s' . The point t , as a vertex of $SPM(s)$, corresponds to a set $M_t(s')$ of vertices in $SPM(s')$. To prove that r_s is an admissible direction for s with respect to t , we need to show that $d(s', t') < d(s, t)$ for any $t' \in M_t(s')$. In the following, we discuss the three cases depending on whether s is in \mathcal{I} , E , or \mathcal{V} .

We remark that the proof would be much simpler if we only considered the “non-degenerate” case whether s is in the interior of a cell of the SPM-equivalence decomposition \mathcal{D}_{spm} (because in that case $M_t(s')$ has only one vertex).

The case $t \in \mathcal{I}$. We first prove the case $t \in \mathcal{I}$. We begin with proving the following claim.

Claim: Suppose we move s to s' infinitesimally along a free direction r ; if there is a point t^ in the interior of the convex hull of the vertices of $U_t(s)$ such that $d(s', v) + |vt^*| \leq d(s, t)$ holds for each $v \in U_t(s)$ and $d(s', v) + |vt^*| < d(s, t)$ holds for at least one vertex $v \in U_t(s)$, then r is in $R(s, t)$.*

We prove the claim as follows. Consider any $t' \in M_t(s')$. To prove $r \in R(s, t)$, it is sufficient to show that $d(s', t') < d(s, t)$.

Since s moves to s' infinitesimally, the distance between t and t' is also infinitesimal. Let H be the convex hull of the vertices of $U_t(s)$. By Observation 1, t is in the interior of H . Hence, t' is also in the interior of H . Further, since $t' \in M_t(s')$ and $U_{t'}(s') \subseteq U_t(s)$, it holds that $d(s', t') = \min_{v \in U_t(s)} (d(s', v) + |vt'|)$.

If $t' = t^*$, because there exists a vertex $v \in U_t(s)$ with $d(s', v) + |vt^*| < d(s, t)$, we can obtain $d(s', t') = \min_{v \in U_t(s)} (d(s', v) + |vt'|) < d(s, t)$, which proves the claim. Below we assume $t' \neq t^*$.

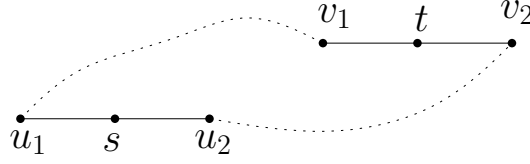


Fig. 6. Suppose s and t have two shortest paths: $\pi_{u_1, v_1}(s, t)$ and $\pi_{u_2, v_2}(s, t)$, with $s \in \overline{u_1 u_2}$ and $t \in \overline{v_1 v_2}$. If s moves towards u_1 and t moves towards v_2 (or s moves towards u_2 and t moves towards v_1) for the same speed, then $d'_{u_i, v_i}(s, t) = 0$ and $d''_{u_i, v_i}(s, t) = 0$ hold for both $i = 1, 2$.

We triangulate H by adding a line segment from t^* to each vertex of H . Let $\triangle v_i v_j t^*$ be a triangle that contains t' , where v_i and v_j are two adjacent vertices of H . Since $t' \neq t^*$, it is easy to see that at least one of $|t' v_i| < |t^* v_i|$ and $|t' v_j| < |t^* v_j|$ must hold. Consequently, we can derive the following

$$\begin{aligned} d(s', t') &= \min_{v \in U_t(s)} (d(s', v) + |vt'|) \leq \min\{d(s', v_i) + |v_i t'|, d(s', v_j) + |v_j t'|\} \\ &< \max\{d(s', v_i) + |v_i t^*|, d(s', v_j) + |v_j t^*|\} \leq d(s, t). \end{aligned}$$

The last inequality is due to the condition in the claim that $d(s', v) + |vt^*| \leq d(s, t)$ holds for each $v \in U_t(s)$. The claim is thus proved.

Now we are back to prove the lemma for the case $t \in \mathcal{I}$. Suppose we move s infinitesimally along r_s with the unit speed to s' , and move t simultaneously along r_t with speed τ ; let t^* be the point where t is located when s arrives at s' . Since $|ss'|$ is infinitesimal, $|tt^*|$ is also infinitesimal. In the following, we will show that t^* satisfies the condition in the above claim, which will lead to the lemma.

For each $v_i \in U_t(s)$, let u_i be the coupled s -pivot of v_i such that either $d'_{u_i, v_i}(s, t) < 0$, or $d'_{u_i, v_i}(s, t) = 0$ and $d''_{u_i, v_i}(s, t) = 0$. Note that $d_{u_i, v_i}(s', t^*)$ is the length of a path from s' to t^* that is the concatenation of $\overline{s' u_i}$, $\pi(u_i, v_i)$, and $\overline{v_i t^*}$. Similarly, $d(s', v_i) + |v_i t^*|$ is also the length of a path from s' to t^* that is the concatenation of $\pi(s', v_i)$, and $\overline{v_i t^*}$. Hence, the following holds

$$d(s', v_i) + |v_i t^*| \leq d_{u_i, v_i}(s', t^*). \quad (2)$$

On the one hand, for any $v_i \in U_t(s)$, since either $d'_{u_i, v_i}(s, t) < 0$ or $d'_{u_i, v_i}(s, t) = 0$, we obtain $d_{u_i, v_i}(s, t) \geq d_{u_i, v_i}(s', t^*)$. Hence, with Inequality (2), we obtain $d(s', v_i) + |v_i t^*| \leq d_{u_i, v_i}(s, t)$ for any $v_i \in U_t(s)$.

On the other hand, since $t \in \mathcal{I}$ and t is a vertex of $SPM(s)$, by Observation 1, $U_t(s)$ has at least three vertices. A vertex v of $U_t(s)$ is called a *special t -pivot* if it has a coupled s -pivot u such that $d'_{u, v}(s, t) = 0$ and $d''_{u, v}(s, t) = 0$ (as shown in [3] this case happens only if r_s is towards u and r_t is leaving v , or r_s is leaving u and r_t is towards v ; e.g., see Fig 6). It was shown in [3] that $U_t(s)$ has at most two special t -pivots (e.g., see Fig 6). Hence, there is at least one vertex $v_i \in U_t(s)$ such that $d'_{u_i, v_i}(s, t) < 0$. This implies that $d(s, t) > d_{u_i, v_i}(s', t^*)$. With Inequality (2), we have $d(s', v_i) + |v_i t^*| < d(s, t)$.

The above proves that t^* satisfies the condition in the claim. Hence, the lemma follows for the case $t \in \mathcal{I}$.

The case $t \in \mathcal{V}$. We proceed on the third case $t \in \mathcal{V}$. Recall the definitions of s' , t' , and $M_t(s')$ in the beginning of the proof of the lemma. Our goal is to prove $d(s, t) > d(s', t')$. Note that since t is a polygon vertex, t' is either t or on one of the two polygon edges of E incident to t [7].

Consider any vertex $v \in U_t(s)$. It has a coupled s -pivot u such that $d'_{u, v}(s, t) < 0$, or $d'_{u, v}(s, t) = 0$ and $d''_{u, v}(s, t) = 0$. In fact, since t does not move (i.e., $\tau = 0$), it is not possible that both $d'_{u, v}(s, t) = 0$ and $d''_{u, v}(s, t) = 0$ hold. Indeed, since $\tau = 0$, according to Equation (1), $d'_{u, v}(s, t) = \cos \gamma_s$ and $d''_{u, v}(s, t) = \frac{\sin^2 \gamma_s}{|su|}$. Both $d'_{u, v}(s, t) = 0$ and $d''_{u, v}(s, t) = 0$ hold if and only if $\cos \gamma_s = \sin \gamma_s = 0$, which is not possible for any angle γ_s . Hence, we obtain $d'_{u, v}(s, t) < 0$. This implies that $d(s, t) = d_{u, v}(s, t) > |s'u| + d(u, v) + |vt|$. Further, since $d(s, t) = |su| + d(u, v) + |vt|$, it holds that $|su| > |s'u|$.

Let v be any vertex of $U_{t'}(s')$. By the definitions of s' and t' , $U_{t'}(s') \subseteq U_t(s)$. Hence, v is in $U_t(s)$. The above shows that v has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$, and $|su| > |s'u|$. Since $v \in U_{t'}(s')$, $\overline{s'u} \cup \pi(u, v) \cup \overline{vt'}$ is a shortest path from s' to t' . Hence, $d(s', t') = |s'u| + d(u, v) + |vt'|$.

If $t' = t$, then since $|su| > |s'u|$, it follows that $d(s', t') = |s'u| + d(u, v) + |vt'| < |su| + d(u, v) + |vt| = d(s, t)$, which proves the lemma.

If $t' \neq t$, as discussed above, t' is on one of the two polygon edges incident to t , which implies that $U_t(s)$ has more than one vertex [7].

We claim that there must exist a vertex $v \in U_t(s)$ such that $|t'v| \leq |tv|$. Indeed, suppose to the contrary that $|t'v| > |tv|$ for every $v \in U_t(s)$. Then, since t' is in $M_t(s')$ and $|ss'|$ is infinitesimal, $|tt'|$ is also infinitesimal. Note that $\pi(s, t)$ cannot be a line segment since otherwise $U_t(s)$ would have only one vertex, incurring contradiction. Also, due to our general position assumption that no three polygon vertices are on the same line, since t is polygon vertex, the last three vertices of any shortest path from s to t are not in the same line. Since $|tt'|$ is infinitesimal, it holds that $d(s, t') = \min_{v \in U_t(s)} (d(s, v) + |vt'|)$ (similar results were also proved in [3]). Since $|t'v| > |tv|$ for every $v \in U_t(s)$, $d(s, t') > \min_{v \in U_t(s)} (d(s, v) + |vt|) = d(s, t)$, which contradicts with that t is a farthest point of s .

In light of the above claim, we assume $|t'v_i| \leq |tv_i|$ for a vertex $v_i \in U_t(s)$. Let u_i be the coupled s -pivot of v_i such that either $d'_{u_i, v_i}(s, t) < 0$ or $d'_{u_i, v_i}(s, t) = 0$. We have shown above that $|su_i| > |s'u_i|$. Note that since $t' \in M_t(s')$, it also holds that $d(s', t') = \min_{v \in U_t(s)} (d(s', v) + |vt'|)$. Based on the above discussion, we can derive the following,

$$\begin{aligned} d(s', t') &= \min_{v \in U_t(s)} (d(s', v) + |vt'|) \leq d(s', v_i) + |v_i t'| \\ &\leq |s'u_i| + d(u_i, v_i) + |v_i t'| < |su_i| + d(u_i, v_i) + |v_i t| = d(s, t). \end{aligned}$$

This proves the lemma for the case $t \in \mathcal{V}$.

The case $t \in \mathbf{E}$. Let e be the polygon edge that contains t . Suppose we move s infinitesimally along r_s with the unit speed to s' and move t simultaneously along r_t with speed τ for the same time to a point t^* (i.e., t^* is the location of t when s arrives at s'). Since $|ss'|$ is infinitesimal, $|tt^*|$ is also infinitesimal. Since r_t is parallel to e and t is in the interior of e , t^* is also in the interior of e . Recall that t corresponds to a set $M_t(s')$ of vertices in $SPM(s')$. Our goal is to prove $d(s', t') < d(s, t)$ for any $t' \in M_t(s')$. Consider any $t' \in M_t(s')$. Below, we prove that $d(s', t') < d(s, t)$.

As shown in [7], t' may be on e or not. In either case, t' is infinitesimally close to t as $|ss'|$ is infinitesimal.

Consider any $v_i \in U_t(s)$. Let u_i be the coupled s -pivot of v such that either $d'_{u_i, v_i}(s, t) < 0$, or $d'_{u_i, v_i}(s, t) = 0$ and $d''_{u_i, v_i}(s, t) = 0$. Hence, $d_{u_i, v_i}(s, t) \geq d_{u_i, v_i}(s', t^*)$. By Observation 1, at least one vertex of $U_t(s)$ must be in the open half-plane bounded by the supporting line of e and containing the interior of \mathcal{P} in the small neighborhood of e ; let v_j be such a vertex. Since t moves along the direction r_t , which is parallel to e , according to Equation (1), it is not possible that $d''_{u_j, v_j}(s, t) = 0$. This implies that $d'_{u_j, v_j}(s, t) < 0$. Consequently, we obtain $d_{u_j, v_j}(s, t) > d_{u_j, v_j}(s', t^*)$.

Next we prove the following *claim*: For any point q on e that is infinitesimally close to t , it holds that $\min_{v_i \in U_t(s)} (|s'u_i| + d(u_i, v_i) + |v_i q|) < d(s, t)$.

Indeed, if $q = t^*$, then we have $\min_{v_i \in U_t(s)} (|s'u_i| + d(u_i, v_i) + |v_i q|) \leq |s'u_j| + d(u_j, v_j) + |v_j q| = d_{u_j, v_j}(s', t^*) < d_{u_j, v_j}(s, t) = d(s, t)$.

Next we assume $q \neq t^*$. By Observation 1, $U_t(s)$ has at least one vertex in each of the two open half-planes bounded by l_t , where l_t is the line through s and perpendicular to e . Without loss of generality, we assume e is horizontal (e.g., see Fig. 7). Hence, there is a vertex $v' \in U_t(s)$ strictly to the left of l_t and a vertex $v'' \in U_t(s)$ strictly to the right of l_t . Since both q and t^* are infinitesimally close to t , v' (resp., v'') is strictly to the left (resp., right) of both q and t^* . Without loss of generality, we

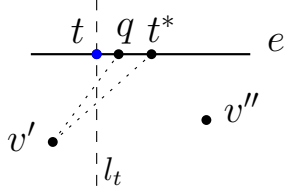


Fig. 7. Since v' is to the left of both q and t^* and q is to the left of t^* , it holds that $|v'q| < |v't^*|$.

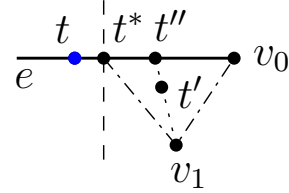


Fig. 8. Illustrating the proof: $|v_1 t^*| > |v_1 t''| > |v_1 t'|$.

assume q is to the left of t^* . Then, we have $|qv'| < |t^*v'|$. Let v' be v_k for some index k . Consequently, $\min_{v_i \in U_t(s)} (|s'u_i| + d(u_i, v_i) + |v_i q|) \leq |s'u_k| + d(u_k, v_k) + |v_k q| < |s'u_k| + d(u_k, v_k) + |v_k t^*| = d_{u_k, v_k}(s', t^*) \leq d_{u_k, v_k}(s, t) = d(s, t)$.

Therefore, the above claim is proved.

Now we are back to our original problem for proving $d(s', t') < d(s, t)$. Depending on whether t' is on e or not, there are two cases.

If t' is on e , then since t' is infinitesimally close to t , then by the above claim, it holds that $\min_{v_i \in U_t(s)} (|s'u_i| + d(u_i, v_i) + |v_i t'|) < d(s, t)$. Note that since t' is in $M_t(s')$, $d(s', t') = \min_{v_i \in U_t(s)} (|s'u_i| + d(u_i, v_i) + |v_i t'|)$. Hence, we obtain $d(s', t') < d(s, t)$.

If t' is not on e , the proof is somewhat similar in spirit to the case $t \in \mathcal{I}$.

We begin with proving an *observation* that there must be a vertex $v \in U_t(s)$ such that $|vt'| < |vt^*|$. Without loss of generality, we assume e is horizontal. All vertices of $U_t(s)$ are in one of the closed half-planes bounded by l_e , where l_e is the horizontal line containing e . Without loss of generality, we assume all vertices of $U_t(s)$ are below or on the line l_e , i.e., they are in the closed half-plane bounded by l_e from above (let h denote the half-plane). Since t' is not on e and t' is infinitesimally close to t , t' is strictly below l_e . Next, we do a “triangulation” around the point t^* . Imagine that we rotate a rightwards ray originated from t^* clockwise to sweep the half-plane h , and let v_1, v_2, \dots, v_m be the vertices of $U_t(s)$ hit by our sweeping ray in order. Note that v_1 and v_m may be on e . If v_1 is not on e , let v_0 be the right endpoint of e . If v_m is not on e , let v_{m+1} be the left endpoint of e . Since t' is infinitesimally close to t and t' is in h , t' must be in one of the triangles $\triangle t^* v_i v_{i+1}$ for $0 \leq i \leq m$.

Suppose t' is in $\triangle t^* v_i v_{i+1}$ for some $1 \leq i \leq m-1$. Then, v_i and v_{i+1} are both from $U_t(s)$. Since t' is in $\triangle t^* v_i v_{i+1}$, one of $|t'v_i| < |t^*v_i|$ and $|t'v_{i+1}| < |t^*v_{i+1}|$ must hold, and this proves the above observation.

If t' is in $\triangle t^* v_i v_{i+1}$ for $i = 0$ or m , then one of v_i and v_{i+1} is not in $U_t(s)$. So we cannot use the same argument as above. In the following, we only prove the case for $i = 0$, and the other case is similar. When t' is in $\triangle t^* v_0 v_1$, v_0 is the right endpoint of e and v_1 is not on e (e.g., see Fig. 8). In the following, we show that $|v_1 t'| < |v_1 t^*|$, which will prove the observation.

By Observation 1, $U_t(s)$ has at least one vertex strictly to right of the vertical line through t . Since $|tt^*|$ is infinitesimal, $U_t(s)$ has at least one vertex strictly to the right of the vertical line l_{t^*} through t^* as well. By the definition of v_1 , v_1 must be strictly to the right of l_{t^*} . Hence, the slope of the line through v_1 and t^* is strictly negative. Recall that t' is in $\triangle t^* v_0 v_1$. If t' is on $\overline{v_1 t^*}$, then since $t' \neq t^*$ (due to $t' \notin e$), we have $|v_1 t^*| > |v_1 t'|$. Otherwise, we extend $\overline{v_1 t'}$ until it hits $\overline{v_0 t^*}$ at a point t'' , which is strictly to the right of t^* . Since both t^* and t' are infinitesimally close to t , $|t^* t'|$ is also infinitesimal and thus $|t^* t''|$ is infinitesimal as well. Since the slope of $\overline{v_1 t^*}$ is strictly negative and t'' is strictly to the right of t^* , we obtain $|v_1 t^*| > |v_1 t''| > |v_1 t'|$.

The above proves the observation.

In light of the above observation, we assume $|t'v_i| < |t^*v_i|$ for a vertex $v_i \in U_t(s)$. Let u_i be the coupled s -pivot of v_i such that either $d'_{u_i, v_i}(s, t) < 0$, or $d'_{u_i, v_i}(s, t) = 0$. Note that $d_{u_i, v_i}(s, t) \geq d_{u_i, v_i}(s', t^*)$. Since t' is in $M_t(s')$, $d(s', t') = \min_{v \in U_t(s)} (d(s', v) + |vt'|)$. Based on the above discussion,

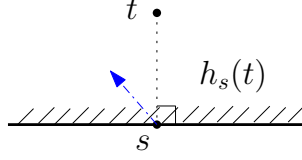


Fig. 9. Illustrating Observation 2: The blue direction is towards the interior of $h_s(t)$ and thus is in $R(s, t)$.

we derive the following

$$\begin{aligned} d(s', t') &= \min_{v \in U_t(s)} (d(s', v) + |\overline{vt'}|) \leq d(s', v_i) + |v_i t'| < d(s', v_i) + |v_i t^*| \\ &\leq |s' u_i| + d(u_i, v_i) + |v_i t^*| = d_{u_i, v_i}(s', t^*) \leq d_{u_i, v_i}(s, t) = d(s, t). \end{aligned}$$

This proves that $d(s', t') < d(s, t)$. This finishes the proof for the case $t \in E$.

The lemma is thus proved for all three cases. \square

Lemma 2 is on the case where s is not visible to t . If s is visible to t , the result is trivial, as shown in Observation 2.

Observation 2 Suppose t is a farthest point of s and s is visible to t . Then t must be a polygon vertex of \mathcal{V} . Further, a free direction r_s of s is in $R(s, t)$ if and only if r_s is towards the interior of $h_s(t)$, where $h_s(t)$ is the open half-plane containing t and bounded by the line through s and perpendicular to \overline{st} (e.g., see Fig. 9).

Proof: Since s is visible to t , \overline{st} is the only shortest path from s to t . As t is a vertex of $SPM(s)$, t cannot be in \mathcal{I} or E , since otherwise there would be more than one shortest s - t path. Thus, $t \in \mathcal{V}$.

Consider any free direction r_s of s . Suppose we move s infinitesimally along r_s to s' . According to our definition of “visibility”, \overline{st} does not contain any polygon vertex in its interior. Since $|ss'|$ is infinitesimal, s' is also visible to t . Therefore, the point t , as a vertex of $SPM(s)$, corresponds to a vertex of $SPM(s')$ that is t itself. Hence, r_s is in $R(s, t)$ if and only if $|s't| < |st|$. Clearly, $|s't| < |st|$ if and only if r_s is towards the interior of the open half-plane $h_s(t)$. \square

By Observation 2, if s is visible to t , then the range $R(s, t)$ is the intersection of the free direction range $R_f(s)$ and an open range of size π delimited by the open half-plane $h_s(t)$.

The next lemma is proved by using Lemmas 1 and 2 as well as Observation 2.

Lemma 3. Among all points of \mathcal{P} that have topologically equivalent shortest path maps in \mathcal{P} , there is at most one geodesic center. This implies that each cell or edge of \mathcal{D}_{spm} contains at most one geodesic center in its interior, which further implies that the number of geodesic centers of \mathcal{P} is $O(|\mathcal{D}_{spm}|)$, where $|\mathcal{D}_{spm}|$ is the combinatorial complexity of \mathcal{D}_{spm} .

Proof: Let Q be any set of points of \mathcal{P} that have topologically equivalent shortest path maps in \mathcal{P} . We show that there is at most one geodesic center in Q , which will prove the lemma. Note that any two points of Q must be visible to each other since otherwise their shortest path maps would not be topologically equivalent. Let s be a geodesic center in Q . Let s' be any other point in Q . In the following, we prove that $d_{\max}(s') > d_{\max}(s)$, which implies that s' cannot be a geodesic center, and thus the lemma will be proved.

Consider the direction r_s of moving s towards s' . Since s is visible to s' , r_s is a free direction. Since s is a geodesic center, by Lemma 1, $R(s)$ is empty. Thus, s has a farthest point t such that $r_s \notin R(s, t)$. Because t is a farthest point of s , $d(s, t) = d_{\max}(s)$.

If s is visible to t , by Observation 2, r_s is not towards the interior of the open half-plane $h_s(t)$. Hence, $|s't| > |st|$, and thus $d_{\max}(s') \geq d(s', t) \geq |s't| > |st| = d(s, t) = d_{\max}(s)$.

In the following, we assume that s is not visible to t . Depending on whether t is in \mathcal{V} , E , or \mathcal{I} , there are three cases.

The case $t \in \mathcal{V}$. Suppose $t \in \mathcal{V}$. Due to $r_s \notin R(s, t)$, by Lemma 2(3), if we move s along r_s with unit speed, there exists a vertex $v \in U_t(s)$ such that either $d'_{u,v}(s, t) > 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, v) \neq 0$, for any coupled s -pivot u of v . Let u be any coupled s -pivot of v .

Note that $\overline{su} \cup \pi(u, v) \cup \overline{vt}$ is a shortest path from s to t , whose length is $d_{u,v}(s, t)$. Since $SPM(s)$ and $SPM(s')$ are topologically equivalent and t is a polygon vertex, $\overline{s'u} \cup \pi(u, v) \cup \overline{vt}$ is a shortest path from s' to t , whose length is $d_{u,v}(s', t)$. In the following, we show that $d_{u,v}(s', t) > d_{u,v}(s, t)$.

Indeed, as s moves towards s' along r_s with unit speed, recall that the second derivative $d''_{u,v}(s, t) \geq 0$ always holds. Hence, if $d'_{u,v}(s, t) > 0$, then during the movement of s , it always holds that $d'_{u,v}(s, t) > 0$. This implies that $d_{u,v}(s', t) > d_{u,v}(s, t)$. Similarly, if $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, t) \neq 0$, then since $d''_{u,v}(s, t) \geq 0$, we obtain $d''_{u,v}(s, t) > 0$. Consequently, as s moves towards s' along r_s with unit speed, except for the starting moment, it holds that $d'_{u,v}(s, t) > 0$ and $d''_{u,v}(s, t) \geq 0$. Thus, $d_{u,v}(s', t) > d_{u,v}(s, t)$.

The above proves that $d(s', t) = d_{u,v}(s', t) > d_{u,v}(s, t) = d(s, t)$. Therefore, we obtain $d_{\max}(s') > d_{\max}(s)$ since $d_{\max}(s') \geq d(s', t) > d(s, t) = d_{\max}(s)$.

The case $t \in E$. If $t \in E$, let e denote the polygon edge of E that contains t . Due to $r_s \notin R(s, t)$, by Lemma 2(2), if we move s along r_s with unit speed and move t along e with any speed $\tau \geq 0$, there exists a vertex $v \in U_t(s)$ such that either $d'_{u,v}(s, t) > 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, v) \neq 0$, for any coupled s -pivot u of v . Let u be any coupled s -pivot of v .

Note that $\overline{su} \cup \pi(u, v) \cup \overline{vt}$ is a shortest path from s to t , whose length is $d_{u,v}(s, t)$. Since $SPM(s)$ and $SPM(s')$ are topologically equivalent and $t \in e$, the point t , as a vertex of $SPM(s)$, corresponds to one and only one vertex t' in $SPM(s')$ that is also on e . Then, $\overline{s'u} \cup \pi(u, v) \cup \overline{vt'}$ is a shortest path from s' to t' , whose length is $d_{u,v}(s', t')$. In the following, we show that $d_{u,v}(s', t') > d_{u,v}(s, t)$, which will lead to $d_{\max}(s') > d_{\max}(s)$ as $d_{\max}(s') \geq d(s', t') = d_{u,v}(s', t')$ and $d_{\max}(s) = d(s, t) = d_{u,v}(s, t)$.

Suppose we move s along r_s towards s' with unit speed and move t along e towards t' with speed $\tau = |tt'|/|ss'|$. Hence, when s arrives at s' , t arrives at t' simultaneously. By Lemma 2(2), either $d'_{u,v}(s, t) > 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, v) \neq 0$. In either case, by the same analysis as in the above case $t \in \mathcal{V}$, we can show that $d_{u,v}(s', t') > d_{u,v}(s, t)$ and we omit the details.

The case $t \in \mathcal{I}$. The proof for this case is very similar. Due to $r_s \notin R(s, t)$, by Lemma 2(1), if we move s along r_s with unit speed and move t along any free direction with any speed $\tau \geq 0$, there exists a vertex $v \in U_t(s)$ such that either $d'_{u,v}(s, t) > 0$, or $d'_{u,v}(s, t) = 0$ and $d''_{u,v}(s, v) \neq 0$, for any coupled s -pivot u of v .

Since $SPM(s)$ and $SPM(s')$ are topologically equivalent, t corresponds to one and only one vertex t' in $SPM(s')$ that is also in \mathcal{I} . The rest of the argument is exactly the same as that for the case $t \in E$. We can prove that $d_{\max}(s') > d_{\max}(s)$ and we omit the details.

This completes the proof for the lemma. \square

Remark: By extending the proof of Lemma 3, it is possible to obtain a slightly stronger result: Every cell (including its boundary) of \mathcal{D}_{spm} contains at most one geodesic center.

The following corollary can be proved by the same techniques as Lemma 3, and it implies that if t is a farthest point of s , then slightly moving s along a free direction that is not in $R(s, t)$ can never obtain a geodesic center.

Corollary 1. *Suppose t is a farthest point of s . If we move s infinitesimally along a free direction that is not in $R(s, t)$, then $d_{\max}(s)$ will become strictly larger.*

Proof: Let r_s be any free direction that is not in $R(s, t)$. Suppose we move s infinitesimally along r_s to s' . By using exactly the same argument as in the proof of Lemma 3, we can show that $SPM(t)$ has a vertex t' corresponding to t in $SPM(s)$ such that $d(s, t) < d(s', t')$. Therefore, we obtain $d_{\max}(s) = d(s, t) < d(s', t') \leq d_{\max}(s')$, which proves the corollary. \square

So far we have shown that the total number of geodesic centers is bounded by the combinatorial size of \mathcal{D}_{spm} . This result, although it is interesting in its own right, is not quite helpful for computing the geodesic centers. In order to compute candidate points for geodesic centers, we need to find a way to determine the range $R(s, t)$. It turns out that it is sufficient to determine $R(s, t)$ when t is in a non-degenerate position with respect to s in the following sense: Suppose t is a farthest point of s ; we say that t is *non-degenerate* with respect to s if there are exactly three, two, and one shortest s - t paths for t in \mathcal{I} , E , and \mathcal{V} , respectively (by Observation 1, this implies that $|U_t(s)|$ is 3, 2, and 1, respectively for the three cases).

Lemma 2 gives a sufficient condition for a direction in $R(s, t)$. The following lemma gives both a sufficient and a necessary condition for a direction in $R(s, t)$ when t is non-degenerate, and the lemma will be used to explicitly compute the range $R(s, t)$ in Section 4. Note that Observation 2 already gives a way to determine $R(s, t)$ when s is visible to t .

Lemma 4. *Suppose t is a non-degenerate farthest point of s and s is not visible to t . Then, a free direction r_s is in $R(s, t)$ if and only if*

1. *for $t \in \mathcal{I}$, there is a free direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ , each vertex $v \in U_t(s)$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$.*
2. *for $t \in E$, there is a free direction r_t for t that is parallel to the polygon edge containing t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ , each vertex $v \in U_t(s)$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$.*
3. *for $t \in \mathcal{V}$, when we move s along r_s with unit speed, each vertex $v \in U_t(s)$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$.*

Proof: First of all, in any of these three cases, if the condition in the lemma statement holds, by Lemma 2, r_s is in $R(s, t)$. In the following, we prove the other direction of the lemma.

Let r_s be in $R(s, t)$. Suppose we move s along r_s infinitesimally to s' . Since t is non-degenerate with respect to s , the point t , as a vertex of $SPM(s)$, corresponds to one and only one vertex t' in $SPM(s')$ (i.e., $M_t(s') = \{t'\}$) [7]. Due to $r_s \in R(s, t)$, $d(s, t) > d(s', t')$. In the following, we prove the three cases: $t \in \mathcal{I}$, $t \in E$, and $t \in \mathcal{V}$.

The case $t \in \mathcal{I}$. Suppose we move s towards s' with unit speed and move t towards t' with speed $|tt'|/|ss'|$. Then, when s arrives at s' , t arrives at t' simultaneously.

Consider any vertex $v \in U_t(s)$. To prove the lemma for this case, our goal is to show that there exists a coupled s -pivot u of v with $d'_{u,v}(s, t) < 0$.

Let u be any coupled s -pivot of v . Hence, $d(s, t) = d_{u,v}(s, t)$. Since $M_t(s') = \{t'\}$, $\overline{s'u} \cup \pi(u, v) \cup \overline{vt'}$ is a shortest path from s' to t' , and thus $d(s', t') = d_{u,v}(s', t')$. We claim that $d'_{u,v}(s, t) < 0$. Suppose to the contrary that $d'_{u,v}(s, t) \geq 0$. If $d'_{u,v}(s, t) > 0$, then we would obtain $d_{u,v}(s, t) < d_{u,v}(s', t')$, which contradicts with $d(s, t) > d(s', t')$. Similarly, if $d'_{u,v}(s, t) = 0$, since $d''_{u,v}(s, t) \geq 0$, we would obtain $d_{u,v}(s, t) \leq d_{u,v}(s', t')$, which contradicts with $d(s, t) > d(s', t')$. Hence, $d'_{u,v}(s, t) < 0$ is proved.

This proves the lemma for the case $t \in \mathcal{I}$.

The case $t \in E$. Let e be the polygon edge containing t . Then, t' is also on e . Suppose we move s towards s' with unit speed and move t on e towards t' with speed $|tt'|/|ss'|$. Hence, when s arrives at s' , t arrives at t' simultaneously.

Consider any vertex $v \in U_t(s)$. Let u be any coupled s -pivot of v . As in the above case, $d_{u,v}(s, t) = d(s, t)$ and $d_{u,v}(s', t') = d(s', t')$. Using the same analysis as in the above case we can also prove that $d'_{u,v}(s, t) < 0$, which leads to the lemma.

The case $t \in \mathcal{V}$. In this case, since t is a polygon vertex, $t' = t$. Suppose we move s towards s' with unit speed. Consider any vertex $v \in U_t(s)$. Let u be any coupled s -pivot of v . Again, $d_{u,v}(s, t) = d(s, t)$ and $d_{u,v}(s', t') = d(s', t')$. Using the same analysis as in the first case we can also prove that $d'_{u,v}(s, t) < 0$, which leads to the lemma. \square

Lemma 4 will be used to determine the range $R(s, t)$ for a non-degenerate farthest point of s . The details are deferred in Section 4, where we will show that $R(s, t)$ is the intersection of the free direction range $R_f(s)$ and an open range of size π (i.e., the previously mentioned π -range property).

In addition, we present Lemma 5, which will be useful for computing the candidate points in Section 5. If a farthest point t of s is not non-degenerate, then we say that t is *degenerate* (note that s cannot be visible to t in the degenerate case). Lemma 5 provides a sufficient condition for a direction in $R(s, t)$ particularly for a degenerate farthest point t of s .

Lemma 5. *Suppose t is a degenerate farthest point of s .*

1. *For $t \in \mathcal{I}$, a free direction r_s is in $R(s, t)$ if the following conditions are satisfied: (1) there exist three vertices $v_1, v_2, v_3 \in U_t(s)$ such that t is in the interior of the triangle $\triangle v_1 v_2 v_3$ (i.e., $\{v_1, v_2, v_3\}$ satisfies the same condition as $U_t(s)$ in Observation 1(1)); (2) there exists a free direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ , each vertex $v \in \{v_1, v_2, v_3\}$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$.*
2. *For $t \in E$, suppose e is the polygon edge containing t . A free direction r_s is in $R(s, t)$ if the following conditions are satisfied: (1) there exist two vertices $v_1, v_2 \in U_t(s)$ such that $\{v_1, v_2\}$ has one vertex in each of the two open half-planes bounded by the line through t and perpendicular to e , and $\{v_1, v_2\}$ has at least one vertex in the open half-plane bounded by the supporting line of e and containing the interior of \mathcal{P} in the small neighborhood of e (i.e., $\{v_1, v_2\}$ satisfies the same condition as $U_t(s)$ in Observation 1(2)); (2) there is a free direction r_t for t parallel to e with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ , each vertex $v \in \{v_1, v_2\}$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$.*

Proof: The proof uses similar techniques as in the proof of Lemma 2. Indeed, the proof of Lemma 2 mainly relies on Observation 1. Here, for the case $t \in \mathcal{I}$, $\{v_1, v_2, v_3\}$ satisfies Observation 1(1); for the case $t \in E$, $\{v_1, v_2\}$ satisfies Observation 1(2). Therefore, similar techniques as in the proof of Lemma 2 can be used here. We briefly discuss it below.

The case $t \in \mathcal{I}$. We first consider the case $t \in \mathcal{I}$. Let v_1, v_2 , and v_3 be the polygon vertices specified in the lemma statement. For each i with $1 \leq i \leq 3$, let u_i be the coupled s -pivot of v_i with $d'_{u_i, v_i}(s, t) < 0$. Suppose we move s along r_s infinitesimally with unit speed to s' and simultaneously move t along r_t with speed τ to a point t^* (i.e., t^* is the location of t when s arrives at s'). Since $|ss'|$ is infinitesimal, $|tt^*|$ is also infinitesimal. Since t is in the interior of $\triangle v_1 v_2 v_3$, t^* is also in the interior of the triangle. Consider any t' in $M_t(s')$. To prove that r_s is in $R(s, t)$, our goal is to show that $d(s', t') < d(s, t)$.

Since $|ss'|$ is infinitesimal, $|tt^*|$ is also infinitesimal and t' is also in the interior of $\triangle v_1 v_2 v_3$. For each $1 \leq i \leq 3$, since $d'_{u_i, v_i}(s, t) < 0$, it holds that $d_{u_i, v_i}(s, t) > d_{u_i, v_i}(s', t^*)$. The three segments $\overline{t^* v_i}$ for $i = 1, 2, 3$ partition $\triangle v_1 v_2 v_3$ into three smaller triangles and t' must be one of them. Without loss of generality, assume t' is in $\triangle v_1 v_2 t^*$. Hence, at least one of $|t' v_1| < |t^* v_1|$ and $|t' v_2| < |t^* v_2|$ holds. Without loss of generality, we assume the former one holds. Therefore, we can derive $d(s', t') \leq |s' u_1| + d(u_1, v_1) + |v_1 t'| < |s' u_1| + d(u_1, v_1) + |v_1 t^*| = d_{u_1, v_1}(s', t^*) < d_{u_1, v_1}(s, t) = d(s, t)$.

The case $t \in E$. Let v_1 and v_2 be the two polygon vertices specified in the lemma statement. For each i with $1 \leq i \leq 2$, let u_i be the coupled s -pivot of v_i with $d'_{u_i, v_i}(s, t) < 0$. Suppose we move s along r_s infinitesimally with unit speed to s' and simultaneously move t along e with speed τ to a point t^* . Since $|ss'|$ is infinitesimal, $|tt^*|$ is also infinitesimal. Consider any t' in $M_t(s')$. To prove that r_s is in $R(s, t)$, our goal is to show that $d(s', t') < d(s, t)$.

For each $1 \leq i \leq 2$, since $d'_{u_i, v_i}(s, t) < 0$, it holds that $d_{u_i, v_i}(s, t) > d_{u_i, v_i}(s', t^*)$. Since $|tt^*|$ is infinitesimal, $\{v_1, v_2\}$ has one vertex in each of the open half-planes bounded by the line through t^* and perpendicular to e . Also, because at least one vertex of v_1 and v_2 is in the open half-plane bounded by the supporting line of e and containing the interior of \mathcal{P} in the small neighborhood of e , regardless of whether t' is on e or not, we can use the similar approach as in the proof of Lemma 2 (for the case $t \in E$) to show that either $|t'v_1| < |t^*v_1|$ or $|t'v_2| < |t^*v_2|$ holds. Consequently, by the similar argument as the above case for $t \in \mathcal{I}$, we can obtain $d(s', t') < d(s, t)$. \square

4 Determining the Admissible Direction Range $R(s, t)$ and the π -Range Property

In this section, we determine the admissible direction range $R(s, t)$ for any point s and any of its non-degenerate farthest point t . In particular, we will prove the π -range property mentioned in Section 1.

Depending on whether t is in \mathcal{V} , E , and \mathcal{I} , there are three cases. Recall that $R_f(s)$ is the range of all free directions of s . In each case, we will show that $R(s, t)$ is the intersection of $R_f(s)$ and an open range $R_\pi(s, t)$ of size π . We call $R_\pi(s, t)$ the π -range. As will be seen later, the π -range $R_\pi(s, t)$ can be explicitly determined based on the positions of s , t , and the vertices of $U_s(t)$ and $U_t(s)$.

In fact, for each case, we will give more general results that are on shortest path distance functions. These more general results will also be useful for computing the candidate points later in Section 5.

4.1 The Case $t \in \mathcal{V}$

We first discuss the case $t \in \mathcal{V}$. The result is relatively straightforward in this case. If s is visible to t , the π -range $R_\pi(s, t)$ is defined to be the open range of directions delimited by the open half-plane $h_s(t)$ as defined in Observation 2; by Observation 2, $R(s, t) = R_f(s) \cap R_\pi(s, t)$.

In the following, we assume s is not visible to t . We first present a more general result on a shortest path function. Let s and t be any two points in \mathcal{P} such that t is in \mathcal{V} and s is not visible to t . Let $\pi(s, t)$ be any shortest s - t path in \mathcal{P} . Let u and v be the s -pivot and t -pivot in $\pi(s, t)$, respectively. Thus, $d_{u,v}(s, t) = |su| + d(u, v) + |vt|$. Now we consider $d_{u,v}(s, t)$ as a function of s and t in the entire plane \mathbb{R}^2 (not only in \mathcal{P} ; namely, when we move s and t , they are allowed to move outside \mathcal{P} , but the function $d_{u,v}(s, t)$ is always defined as $|su| + d(u, v) + |vt|$, where $d(u, v)$ is a fixed value).

The π -range $R_\pi(s, t)$ is defined with respect to t and the path $\pi(s, t)$ as follows: a direction r_s for s is in $R_\pi(s, t)$ if $d'_{u,v}(s, t) < 0$ when we move s along r_s with unit speed. The following lemma is quite straightforward.

Lemma 6. *The π -range $R_\pi(s, t)$ is exactly the open range of size π delimited by $h_s(u)$, where $h_s(u)$ is the open half-plane containing u and bounded by the line through s and perpendicular to \overline{su} .*

Proof: The analysis is similar to Observation 2. Suppose we move s along a direction r_s with unit speed. Then, $d'_{u,v}(s, t) < 0$ if and only if r_s is towards the interior of $h_s(u)$. \square

Now we are back to our original problem to determine $R(s, t)$ for a non-degenerate farthest point t of s with $t \in \mathcal{V}$. Since t is non-degenerate and t is in \mathcal{V} , there is only one shortest path $\pi(s, t)$ from s to t . We define $R_\pi(s, t)$ as above. Based on Observation 2 and Lemmas 4(3), we have Lemma 7, and thus $R(s, t)$ can be determined by Observation 2 and Lemma 6.

Lemma 7. $R(s, t) = R_f(s) \cap R_\pi(s, t)$.

Proof: If s is visible to t , we have already shown that the lemma is true. Below we assume s is not visible to t .

Let u and v be the s -pivot and t -pivot in $\pi(s, t)$, respectively. Note that v is the only vertex in $U_t(s)$ and u is the only vertex in $U_s(t)$.

1. Consider any direction $r_s \in R(s, t)$, i.e., r_s is an admissible direction for s with respect to t . According to Lemma 4(3), when we move s along r_s with unit speed, it holds that $d'_{u,v}(s, t) < 0$. This implies that r_s is in $R_\pi(s, t)$. Since r_s is in $R(s, t)$, r_s is in $R_f(s)$. Therefore, r_s is in $R_f(s) \cap R_\pi(s, t)$.
2. Consider any direction r_s in $R_f(s) \cap R_\pi(s, t)$. First of all, r_s is a free direction. Since r_s is in $R_\pi(s, t)$, when s moves along r_s with unit speed, $d'_{u,v}(s, t) < 0$. Since v is the only vertex in $U_t(s)$, by Lemma 4(3), r_s is in $R(s, t)$.

The lemma is thus proved. \square

4.2 The Case $t \in E$

The analysis for this case is substantially more complicated than the previous case, although the next case for $t \in \mathcal{I}$ is even more challenging. One may consider the analysis for this case as a “warm-up” for the most general case $t \in \mathcal{I}$.

As in the previous case, we first present a more general result that is on two shortest path distance functions. Let s and t be any two points in \mathcal{P} such that t is in E and there are two shortest s - t paths $\pi_1(s, t)$ and $\pi_2(s, t)$ (this implies that s is not visible to t). Let e be the polygon edge containing t and let $l(e)$ denote the line containing e . For each $i = 1, 2$, let $\pi_i(s, t) = \pi_{u_i, v_i}(s, t)$, i.e., u_i and v_i are the s -pivot and t -pivot of $\pi_i(s, t)$, respectively. We further require the set $\{v_1, v_2\}$ satisfy the same condition as $U_t(s)$ in Observation 1(2), i.e., $\{v_1, v_2\}$ has at least one vertex in the open half-plane bounded by $l(e)$ and containing the interior of \mathcal{P} in the small neighborhood of e , and it has at least one vertex in each of the two open half-planes bounded by the line through t and perpendicular to e . We say that the two shortest paths $\pi_1(s, t)$ and $\pi_2(s, t)$ are *canonical* with respect to s and t if $\{v_1, v_2\}$ satisfies the above condition. In the following, we assume $\pi_1(s, t)$ and $\pi_2(s, t)$ are canonical. Note that the condition implies that $v_1 \neq v_2$. However, $u_1 = u_2$ is possible.

For each $i = 1, 2$, we consider $d_{u_i, v_i}(s, t) = |su_i| + d(u_i, v_i) + |v_i t|$ as a function of $s \in \mathbb{R}^2$ (instead of $s \in \mathcal{P}$ only) and $t \in e$.

In this case, the π -range $R_\pi(s, t)$ of s is defined with respect to t and the two paths $\pi_1(s, t)$ and $\pi_2(s, t)$ as follows: a direction r_s for s is in $R_\pi(s, t)$ if there exists a direction r_t parallel to e for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed $\tau \geq 0$, $d'_{u_i, v_i}(s, t) < 0$ holds for $i = 1, 2$.

In Section 4.1, we showed that the π -range for the case $s \in \mathcal{V}$ is an open range of size π . Here we will show a similar result in Lemma 8 unless a special case happens. Although the analysis in Section 4.1 is quite straightforward, the result here for two functions $d_{u_i, v_i}(s, t)$ with $i = 1, 2$ is somewhat surprising and the proof is substantially more difficult (one could imagine that a similar result for three functions, as shown in Section 4.4, is even more surprising). Before presenting Lemma 8, we introduce some notation.

For any two points p and q in the plane, define \overrightarrow{pq} as the direction from p to q .

Recall that the angle of any direction r is defined to be the angle in $[0, 2\pi)$ counterclockwise from the positive direction of the x -axis. Let α_1 denote the angle of the direction $\overrightarrow{su_1}$, and let α_2 denote the angle of the direction $\overrightarrow{su_2}$ (e.g., see Fig. 10). Note that by our way of defining pivot vertices, $\alpha_1 = \alpha_2$ if and only if $u_1 = u_2$.

Note that v_1 and v_2 are in a closed half-plane bounded by the line $l(e)$. We assign a direction to $l(e)$ such that each of v_1 and v_2 are to the left or on $l(e)$. Define β_i as the smallest angle to rotate $l(e)$ counterclockwise such that the direction of $l(e)$ becomes the same as the direction $\overrightarrow{tv_i}$, for each $i = 1, 2$ (e.g., see Fig. 10). Hence, both β_1 and β_2 are in $[0, \pi]$. Without loss of generality, we assume $\beta_1 \leq \beta_2$ (otherwise the analysis is symmetric). Since $\{v_1, v_2\}$ contains at least one vertex in each of the open half-planes bounded by the line through t and perpendicular to e , we have $\beta_1 \in [0, \pi/2)$ and $\beta_2 \in (\pi/2, \pi]$. Further, since at least one of v_1 and v_2 is not on $l(e)$, it is not possible that both $\beta = 0$ and $\beta = \pi$ hold.

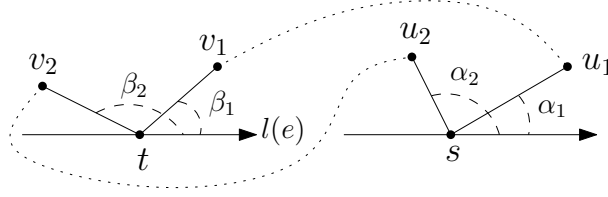


Fig. 10. Illustrating the definitions of the angles.

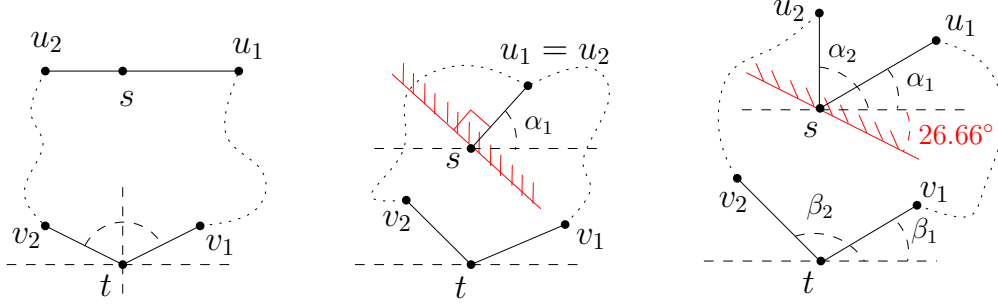


Fig. 11. Illustrating several concrete examples for Lemma 8. Left: the special case (the vertical line through t bisects $\angle v_1tv_2$); in this case, $R_\pi(s, t) = \emptyset$. Middle: the case where $\alpha_1 = \alpha_2$, and thus $\alpha = 0$, $\sin(\alpha) = 0$, and $u_1 = u_2$; in this case, $R_\pi(s, t) = (\alpha_1 - \pi/2, \alpha_1 + \pi/2)$, which is delimited by the open half-plane (marked with red color in the figure) bounded by the line through s and perpendicular to $\overline{su_1}$. Right: the most general case where $\alpha_1 = 30^\circ$, $\alpha_2 = 90^\circ$, $\beta_1 = 30^\circ$, $\beta_2 = 135^\circ$; by calculation, $\lambda \approx 1.3165$, $\arctan(\frac{\lambda}{\sin(\alpha)}) \approx 56.66^\circ$, and thus, $R_\pi(s, t) \approx (\alpha_1 - 56.66^\circ, \alpha_1 + 56.66^\circ) = (-26.66^\circ, 153.34^\circ)$; the open half-plane that delimits $R_\pi(s, t)$ is marked with red color in the figure.

Let $\alpha = \alpha_2 - \alpha_1$. We refer to the case where $\beta_1 + \beta_2 = \pi$ and $\alpha = \pm\pi$ (i.e., α is π or $-\pi$) as the *special case*. In the special case, s is on $\overline{u_1u_2}$ and the vertical line through t and perpendicular to $l(e)$ bisects the angle $\angle v_1tv_2$.

Lemma 8. *The π -range $R_\pi(s, t)$ is determined as follows (e.g., see Fig. 11).*

$$R_\pi(s, t) = \begin{cases} (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}), \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi) & \text{if } \sin(\alpha) > 0, \\ (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)})) & \text{if } \sin(\alpha) < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda > 0, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda < 0, \\ \emptyset & \text{if } \sin(\alpha) = 0 \text{ and } \lambda = 0, \end{cases}$$

where $\lambda = \cos \alpha - \frac{\cos \beta_2}{\cos \beta_1}$. Further, $\alpha = \pm\pi$ and $\beta_1 + \beta_2 = \pi$ (i.e., the special case) if and only if $\sin(\alpha) = 0$ and $\lambda = 0$.

Remark: Unlike Lemma 6 whose geometric intuition is very straightforward, it is not clear to us how to interpret Lemma 8 intuitively.

We defer the proof of Lemma 8 to Section 4.3. According to Lemma 8, if the special case happens, $R_\pi(s, t)$ is empty; otherwise, it is an open range of size exactly π . Since $\alpha = 0$ if and only if $u_1 = u_2$, the case $u_1 = u_2$ is also covered by the lemma.

Now we are back to our original problem to determine the range $R(s, t)$ for a non-degenerate farthest point $t \in E$ of s . By Observation 2, s is not visible to t . Further, s and t have exactly two shortest paths $\pi_1(s, t)$ and $\pi_2(s, t)$. Clearly, by Observation 1(2), the two paths are canonical. Therefore, the π -range $R_\pi(s, t)$ of s with respect to t and the two shortest paths $\pi_1(s, t)$ and $\pi_2(s, t)$ can be determined by Lemma 8.

By using Lemma 4(2), we have the following lemma.

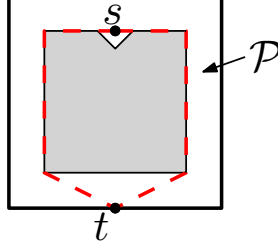


Fig. 12. Illustrating an example in which a geodesic center s is in \mathcal{I} and has only one farthest point t . The polygonal domain \mathcal{P} is between two (very close) concentric squares plus an additional (very small) triangle so that s is in \mathcal{I} . The point s is at the middle of the top edge of the inner square, and t is at the middle of the bottom edge of the outer square. One can verify that s is a geodesic center and t is the only farthest point of s . The two shortest paths from s to t are shown with red dashed segments. Note that the middle point of every edge of the inner square is a geodesic center.

Lemma 9. $R(s, t) = R_\pi(s, t) \cap R_f(s)$.

Proof: For each $i = 1, 2$, let u_i and v_i be the s -pivot and t -pivot of the shortest path $\pi_i(s, t)$, respectively. Since s and t have only two shortest paths $\pi_1(s, t)$ and $\pi_2(s, t)$, $U_t(s) = \{v_1, v_2\}$ and $U_s(t) = \{u_1, u_2\}$. Further, for each $i = 1, 2$, v_i has only one s -pivot, which is u_i .

1. Consider any direction $r_s \in R(s, t)$. Clearly, $r_s \in R_f(s)$. By Lemma 4(2), there exists a direction r_t parallel to e for t with a speed $\tau \geq 0$ such that if we move s along r_s for unit speed and move t along r_t with speed τ , each vertex $v \in U_t(s)$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$. Since $U_t(s) = \{v_1, v_2\}$ and for each $i = 1, 2$, v_i has only one s -pivot u_i , it holds that $d'_{u_i, v_i}(s, t) < 0$. Hence, r_s is in $R_\pi(s, t)$.
2. Consider any r_s in $R_f(s, t) \cap R_\pi(s, t)$. First of all, r_s is a free direction. Since r_s is in $R_\pi(s, t)$, there exists a direction r_t parallel to e for t with a speed $\tau \geq 0$ such that if we move s along r_s for unit speed and move t along r_t with speed τ , $d'_{u_i, v_i}(s, t) < 0$ for each $i = 1, 2$. Since $U_t(s) = \{v_1, v_2\}$, according to Lemma 4(2), r_s is in $R(s, t)$.

The lemma thus follows. \square

Suppose t is the only farthest point of s and t is non-degenerate. According to Lemma 8, if the special case happens, $R_\pi(s, t) = \emptyset$, and thus $R(s, t) = \emptyset$ by Lemma 9. By Corollary 1, whenever we move s along any free direction infinitesimally, the value $d_{\max}(s)$ will be strictly increasing. Therefore, it is possible that the point s , which is in \mathcal{I} and has only one farthest point, is a geodesic center. It is not difficult to construct such an example by following the left figure of Fig. 11; e.g., see Fig. 12. Hence, we have the following corollary.

Corollary 2. *It is possible that a geodesic center is in \mathcal{I} and has only one farthest point.*

4.3 Proof of Lemma 8

Consider any direction r_s for moving s . Let θ_s denote the angle of r_s . Recall that the moving direction r_t for t is parallel to e and we have assigned a direction to the line $l(e)$. Let θ_t denote the smallest angle to rotate $l(e)$ such that $l(e)$ becomes the same direction as r_t (thus the definition of θ_t is “consistent” with the definitions of β_1 and β_2). Since r_t is parallel to e , θ_t is either 0 or π .

According to Equation (1), we can obtain the derivatives of the two functions $d_{u_1, v_1}(s, t)$ and $d_{u_2, v_2}(s, t)$ as follows

$$\begin{cases} d'_{u_1, v_1}(s, t) = -\cos(\alpha_1 - \theta_s) - \tau \cdot \cos(\beta_1 - \theta_t), \\ d'_{u_2, v_2}(s, t) = -\cos(\alpha_2 - \theta_s) - \tau \cdot \cos(\beta_2 - \theta_t). \end{cases} \quad (3)$$

Therefore, for each $i = 1, 2$, $d'_{u_i, v_i}(s, t)$ is a function of θ_s , θ_t , and τ . In order to simplify our proof for Lemma 8, we first give the following lemma.

Lemma 10. *A direction r_s is in $R_\pi(s, t)$ if and only if there exist $\theta_t \in \{0, \pi\}$ and $\tau \geq 0$ such that $d'_{u_1, v_1}(s, t) = 0$ and $d'_{u_2, v_2}(s, t) < 0$ (the same also holds if we exchange the indices 1 and 2).*

Proof: Given any direction r_s for s , the angle θ_s is fixed, and thus for each $i = 1, 2$, $d'_{u_i, v_i}(s, t)$ is a function of θ_t and τ . In the following, we will use $d'_i(\theta_t, \tau)$ to represent $d'_{u_i, v_i}(s, t)$ for each $i = 1, 2$.

We first prove one direction of the lemma. Assume $d'_1(\theta_t, \tau) = 0$ and $d'_2(\theta_t, \tau) < 0$ for some $\theta_t \in \{0, \pi\}$ and $\tau \geq 0$. Our goal is to show that r_s is in $R_\pi(s, t)$. To this end, it is sufficient to find another pair (θ'_t, τ') with $\theta'_t \in \{0, \pi\}$ and $\tau' \geq 0$ such that $d'_1(\theta'_t, \tau') < 0$ and $d'_2(\theta'_t, \tau') < 0$.

Recall that neither β_1 nor β_2 can be $\pi/2$. Since θ_t is either 0 or π , $\beta_1 - \theta_t \neq \pm\pi/2$, and thus $\cos(\beta_1 - \theta_t) \neq 0$. Since $d'_1(\theta_t, \tau) = 0$ and $d'_2(\theta_t, \tau) < 0$, if $\tau > 0$, regardless of whether $\cos(\beta_1 - \theta_t)$ is positive or negative, we can always change τ infinitesimally to a new value $\tau' > 0$ such that $d'_1(\theta_t, \tau') < 0$ and $d'_2(\theta_t, \tau') < 0$.

If $\tau = 0$, then by Equation (3), $-\cos(\alpha_1 - \theta_s) = 0$ and $-\cos(\alpha_2 - \theta_s) < 0$. We let τ' be an infinitesimally small positive value and let θ'_t be 0. Since $\beta_1 \in [0, \pi/2)$, we have $\cos(\beta_1 - \theta'_t) > 0$, and thus $d'_1(\theta'_t, \tau') < 0$ due to $-\cos(\alpha_1 - \theta_s) = 0$. Further, since τ' is an infinitesimal small positive value and $-\cos(\alpha_2 - \theta_s) < 0$, we have $d'_2(\theta'_t, \tau') < 0$.

This proves that r_s must be in $R_\pi(s, t)$.

We proceed to prove the other direction of the lemma. Assume r_s is in $R_\pi(s, t)$. By the definition of $R_\pi(s, t)$, there exist $\theta_t \in \{0, \pi\}$ and $\tau \geq 0$ such that $d'_1(\theta_t, \tau) < 0$ and $d'_2(\theta_t, \tau) < 0$. Our goal is to find another pair (θ'_t, τ') with $\theta'_t \in \{0, \pi\}$ and $\tau' \geq 0$ such that $d'_1(\theta'_t, \tau') = 0$ and $d'_2(\theta'_t, \tau') < 0$.

Recall that $\cos(\beta_1 - \theta_t) \neq 0$. Depending on whether $\cos(\beta_1 - \theta_t)$ is positive or negative, there are two cases.

If $\cos(\beta_1 - \theta_t) < 0$, then $\cos(\beta_2 - \theta_t) > 0$ because $\beta_1 \in [0, \pi/2)$, $\beta_2 \in (\pi/2, \pi]$, and $\theta_t \in \{0, \pi\}$. Since $\tau \geq 0$, if we increase the value τ , $d'_1(\theta_t, \tau)$ will strictly increase and $d'_2(\theta_t, \tau)$ will strictly decrease. Hence, if we keep increasing τ , there must be a moment when $d'_1(\theta_t, \tau) = 0$ and $d'_2(\theta_t, \tau) < 0$. We are done with the proof.

If $\cos(\beta_1 - \theta_t) > 0$, then $\cos(\beta_2 - \theta_t) < 0$. Depending on whether $\tau = 0$, there are two subcases.

- If $\tau = 0$, let $\theta'_t = (\theta_t + \pi \bmod 2\pi)$ (i.e., we reverse the moving direction of t). Consequently, we obtain $\cos(\beta_1 - \theta'_t) < 0$ and $\cos(\beta_2 - \theta'_t) > 0$. Then, we can use the same approach as above, i.e., keep increasing τ until $d'_1(\theta'_t, \tau) = 0$ and $d'_2(\theta'_t, \tau) < 0$.
- If $\tau > 0$, then if we decrease τ , $d'_1(\theta_t, \tau)$ increases and $d'_2(\theta_t, \tau)$ decreases. We keep decreasing the value τ until one of the two events happens: $\tau = 0$ and $d'_1(\theta_t, \tau) = 0$.

Whichever event happens first, it always holds that $d'_2(\theta_t, \tau) < 0$. If $d'_1(\theta_t, \tau) = 0$ happens first (or both events happen simultaneously), we are done with the proof. Otherwise, we obtain $\tau = 0$ and both $d'_1(\theta_t, \tau)$ and $d'_2(\theta_t, \tau)$ are negative. Then, we can use the same approach as the above case for $\tau = 0$ to prove.

This completes the proof for the lemma. \square

To simplify the notation, let $w_1 = -d'_{u_1, v_1}(s, t)$ and $w_2 = -d'_{u_2, v_2}(s, t)$. Once r_s is fixed (and thus θ_s is fixed), both w_1 and w_2 are implicitly considered as functions of $\theta_t \in \{0, \pi\}$ and $\tau \geq 0$. By Lemma 10, $R_\pi(s, t)$ consists of all directions r_s for s such that there exist $\theta_t \in \{0, \pi\}$ and $\tau \geq 0$ with $w_1 = 0$ and $w_2 > 0$.

Let $x = \alpha_1 - \theta_s$. Recall that $\alpha = \alpha_2 - \alpha_1$. Then, $\alpha_2 - \theta_s = x + \alpha$. Thus, we have

$$\begin{cases} w_1 = \cos(x) + \tau \cdot \cos(\beta_1 - \theta_t), \\ w_2 = \cos(x + \alpha) + \tau \cdot \cos(\beta_2 - \theta_t). \end{cases}$$

First of all, let $w_1 = 0$. As shown in the proof of Lemma 10, since $\beta_1 \in [0, \pi/2)$ and $\theta_t \in \{0, \pi\}$, it holds that $\cos(\beta_1 - \theta_t) \neq 0$. Consequently, we have

$$\tau = -\frac{\cos(x)}{\cos(\beta_1 - \theta_t)}.$$

Since $\tau \geq 0$ and $\beta_1 \in [0, \pi/2)$, we obtain the following: if $\cos(x) > 0$, then $\theta_t = \pi$; if $\cos(x) < 0$, then $\theta_t = 0$; if $\cos(x) = 0$, then θ_t can be either 0 or π .

By substituting τ with $-\frac{\cos(x)}{\cos(\beta_1 - \theta_t)}$ in w_2 and using the angle sum identities of trigonometric functions, we have

$$\begin{aligned} w_2 &= \cos(x + \alpha) + \tau \cdot \cos(\beta_2 - \theta_t) \\ &= \cos(x) \cdot \cos(\alpha) - \sin(x) \cdot \sin(\alpha) - \frac{\cos(x)}{\cos(\beta_1 - \theta_t)} \cdot \cos(\beta_2 - \theta_t) \\ &= \cos(x) \cdot \cos(\alpha) - \sin(x) \cdot \sin(\alpha) - \cos(x) \cdot \frac{\cos(\beta_2)}{\cos(\beta_1)}. \end{aligned}$$

The last equality in the above equation is because regardless of whether θ_t is 0 or π , it always holds that $\frac{\cos(\beta_2 - \theta_t)}{\cos(\beta_1 - \theta_t)} = \frac{\cos(\beta_2)}{\cos(\beta_1)}$. Recall that $\lambda = \cos(\alpha) - \frac{\cos(\beta_2)}{\cos(\beta_1)}$ in the statement of Lemma 8. Hence, $w_2 = \cos(x) \cdot \lambda - \sin(x) \cdot \sin(\alpha)$. Therefore, $w_2 > 0$ is equivalent to $\cos(x) \cdot \lambda > \sin(x) \cdot \sin(\alpha)$.

The above discussion shows the following observation.

Observation 3 $w_1 = 0$ and $w_2 > 0$ if and only if the following holds:

$$\cos(x) \cdot \lambda > \sin(x) \cdot \sin(\alpha). \quad (4)$$

Depending on whether $\sin(\alpha)$ is positive, negative, or zero, there are three cases.

1. If $\sin(\alpha) > 0$, depending on whether $\cos(x)$ is positive, negative, or zero, there are further three subcases.
 - (a) If $\cos(x) > 0$, then $x \in (-\pi/2, \pi/2)$ and Inequality (4) is equivalent to $\tan(x) < \frac{\lambda}{\sin(\alpha)}$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (-\pi/2, \arctan(\frac{\lambda}{\sin(\alpha)}))$.
 - (b) If $\cos(x) < 0$, then $x \in (-3\pi/2, -\pi/2)$ and Inequality (4) is equivalent to $\tan(x) > \frac{\lambda}{\sin(\alpha)}$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, -\pi/2)$.
 - (c) If $\cos(x) = 0$, then $x = \pm\frac{\pi}{2}$ and Inequality (4) is equivalent to $\sin(x) < 0$, which further implies that x must be $-\pi/2$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x = -\pi/2$.

Combining the above discussions for the case $\sin(\alpha) > 0$, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, \arctan(\frac{\lambda}{\sin(\alpha)}))$.

2. If $\sin(\alpha) < 0$, the analysis is symmetric. Depending on whether $\cos(x)$ is positive, negative, or zero, there are further three subcases.
 - (a) If $\cos(x) > 0$, then $x \in (-\pi/2, \pi/2)$ and Inequality (4) is equivalent to $\tan(x) > \frac{\lambda}{\sin(\alpha)}$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}), \pi/2)$.
 - (b) If $\cos(x) < 0$, then $x \in (\pi/2, 3\pi/2)$ and Inequality (4) is equivalent to $\tan(x) < \frac{\lambda}{\sin(\alpha)}$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (\pi/2, \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi)$.
 - (c) If $\cos(x) = 0$, then $x = \pm\frac{\pi}{2}$ and Inequality (4) is equivalent to $\sin(x) > 0$, which further implies that x must be $\pi/2$. Thus, $w_1 = 0$ and $w_2 > 0$ if and only if $x = \pi/2$.

Combining the above discussions for the case $\sin(\alpha) < 0$, $w_1 = 0$ and $w_2 > 0$ if and only if $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}), \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi)$.

3. If $\sin(\alpha) = 0$, then Inequality (4) is equivalent to $\cos(x) \cdot \lambda > 0$. Depending on whether λ is positive, negative, or zero, there are further three subcases.
- (a) If $\lambda > 0$, then Inequality (4) is equivalent to $\cos(x) > 0$, implying that $x \in (-\pi/2, \pi/2)$.
 - (b) If $\lambda < 0$, then Inequality (4) is equivalent to $\cos(x) < 0$, implying that $x \in (\pi/2, 3\pi/2)$.
 - (c) If $\lambda = 0$, then Inequality (4) is equivalent to $0 < 0$, which is not possible for any x .

As a summary, $w_1 = 0$ and $w_2 > 0$ if and only if: for $\sin(\alpha) > 0$, $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, \arctan(\frac{\lambda}{\sin(\alpha)}))$; for $\sin(\alpha) < 0$, $x \in (\arctan(\frac{\lambda}{\sin(\alpha)}), \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi)$; for $\sin(\alpha) = 0$ and $\lambda > 0$, $x \in (-\pi/2, \pi/2)$; for $\sin(\alpha) = 0$ and $\lambda < 0$, $x \in (\pi/2, 3\pi/2)$; for $\sin(\alpha) = 0$ and $\lambda = 0$, $x \in \emptyset$.

Recall that $x = \alpha_1 - \theta_s$. By Lemma 10, we obtain the π -range $R_\pi(s, t)$ as follows.

$$R_\pi(s, t) = \begin{cases} (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}), \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi) & \text{if } \sin(\alpha) > 0, \\ (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)})) & \text{if } \sin(\alpha) < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda > 0, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda < 0, \\ \emptyset & \text{if } \sin(\alpha) = 0 \text{ and } \lambda = 0. \end{cases}$$

We complete the proof of Lemma 8 by showing the following claim: $\alpha = \pm\pi$ and $\beta_1 + \beta_2 = \pi$ if and only if $\sin(\alpha) = 0$ and $\lambda = 0$. We prove the claim below.

Assume $\alpha = \pm\pi$ and $\beta_1 + \beta_2 = \pi$. Then, $\sin(\alpha) = 0$, and $\lambda = \cos(\alpha) - \frac{\cos(\beta_2)}{\cos(\beta_1)} = -1 - \frac{\cos(\beta_2)}{\cos(\beta_1)}$. Since $\beta_1 + \beta_2 = \pi$, $\cos(\beta_1) = -\cos(\beta_2)$. Hence, $\lambda = 0$.

On the other hand, assume $\sin(\alpha) = 0$ and $\lambda = 0$. Then, α is 0 or $\pm\pi$. We claim that α cannot be zero. Indeed, suppose to the contrary that $\alpha = 0$. Since $\beta_1 \in [0, \pi/2)$ and $\beta_2 \in (\pi/2, \pi]$, it always holds that $\frac{\cos(\beta_2)}{\cos(\beta_1)} < 0$. Hence, $\lambda = \cos(\alpha) - \frac{\cos(\beta_2)}{\cos(\beta_1)} = 1 - \frac{\cos(\beta_2)}{\cos(\beta_1)}$, which is always positive, contradicting with $\lambda = 0$.

The above proves that $\alpha = \pm\pi$. Then, we have $\lambda = -1 - \frac{\cos(\beta_2)}{\cos(\beta_1)}$. Due to $\lambda = 0$, it holds that $\cos(\beta_2) = -\cos(\beta_1)$, which implies that $\beta_1 + \beta_2 = \pi$ since $\beta_1 \in [0, \pi/2)$ and $\beta_2 \in (\pi/2, \pi]$.

4.4 The Case $t \in \mathcal{I}$

The analysis for this case is substantially more difficult than the case $t \in E$. As before, we first present a more general result that is on three shortest path distance functions.

Let s and t be any two points in \mathcal{P} such that t is in \mathcal{I} and there are three shortest s - t paths $\pi_1(s, t)$, $\pi_2(s, t)$, and $\pi_3(s, t)$ (this implies that s is not visible to t). For each $i = 1, 2, 3$, let $\pi_i(s, t) = \pi_{u_i, v_i}(s, t)$, i.e., u_i and v_i are the s -pivot and t -pivot of $\pi_i(s, t)$, respectively. We say that the three paths are *canonical* with respect to s and t if they have the following two properties.

1. t is in the interior of the triangle $\triangle v_1 v_2 v_3$.
2. Suppose we reorder the indices such that v_1, v_2 , and v_3 are *clockwise* around t , then u_1, u_2 , and u_3 are *counterclockwise* around s (e.g., see Fig. 2).

The above first property implies that v_1, v_2 , and v_3 are distinct, but this may not be true for u_1, u_2, u_3 . In the following, we assume that the three shortest paths $\pi_i(s, t)$ with $1 \leq i \leq 3$ are canonical, and we reorder the indices such that v_1, v_2 , and v_3 are clockwise around t and u_1, u_2 , and u_3 are counterclockwise around s . For each $i = 1, 2, 3$, we consider $d_{u_i, v_i}(s, t) = |su_i| + d(u_i, v_i) + |v_i t|$ as a function of $s \in \mathbb{R}^2$ and $t \in \mathbb{R}^2$.

In this case, the π -range $R_\pi(s, t)$ of s is defined with respect to t and the three paths $\pi_i(s, t)$ for $i = 1, 2, 3$ as follows: a direction r_s for s is in $R_\pi(s, t)$ if there exists a direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ , $d'_{u_i, v_i} < 0$ holds for $i = 1, 2, 3$.

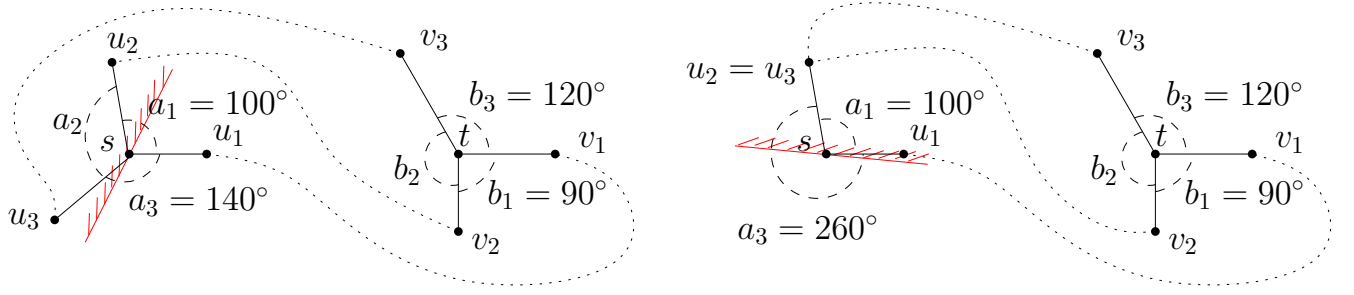


Fig. 13. Illustrating two concrete examples for Lemma 11. Left: The sizes of the angles of a_i and b_i for $1 \leq i \leq 3$ are already shown in the figure with $\alpha_1 = 0$. By calculation, $\delta \approx 0.2426$, $\delta_1 \approx -0.1736$, $\delta_2 \approx 0.3072$, $\arctan(\frac{\delta_1 - \delta_2}{\delta}) \approx -63.23^\circ$, and thus $R_\pi(s, t) \approx (\alpha_1 + 63.23^\circ, \alpha_1 + 63.23^\circ + 180^\circ) = (63.23^\circ, 243.23^\circ)$. Right: a case where $\alpha_2 = \alpha_3$ with $\alpha_1 = 0$. Thus, $a_2 = 0$ and $u_2 = u_3$. The sizes of other angles are already shown in the figure. By calculation, $\delta \approx 2.1220$, $\delta_1 \approx -0.1736$, $\delta_2 \approx -0.3768$, $\arctan(\frac{\delta_1 - \delta_2}{\delta}) \approx 5.47^\circ$, and thus $R_\pi(s, t) \approx (\alpha_1 - 5.47^\circ, \alpha_1 - 5.47^\circ + 180^\circ) = (-5.47^\circ, 174.53^\circ)$. The open half-planes that delimit $R_\pi(s, t)$ in both examples are marked with red color.

As Lemma 8 in the previous cases, we will have a similar lemma (Lemma 11), which says that unless a special case happens the range $R_\pi(s, t)$ is an open range of size exactly π . The proof is much more challenging. Before presenting Lemma 11, we introduce some notation.

Recall the definitions of the angles of directions. For each $i = 1, 2, 3$, let β_i denote the angle of the direction $\overrightarrow{tv_i}$ (i.e., the angle of $\overrightarrow{tv_i}$ counterclockwise from the positive x -axis). Further, we define three angles b_i for $i = 1, 2, 3$ as follows (e.g., see Fig. 2). Define b_1 as the smallest angle we need to rotate the direction $\overrightarrow{tv_1}$ clockwise to $\overrightarrow{tv_2}$; define b_2 as the smallest angle we need to rotate the direction $\overrightarrow{tv_2}$ clockwise to $\overrightarrow{tv_3}$; define b_3 as the smallest angle we need to rotate the direction $\overrightarrow{tv_3}$ clockwise to $\overrightarrow{tv_1}$.

For any two angles α' and α'' , we use $\alpha' \equiv \alpha''$ to denote $\alpha' = \alpha'' \pmod{2\pi}$.

It is easy to see that $b_1 \equiv \beta_1 - \beta_2$, $b_2 \equiv \beta_2 - \beta_3$, and $b_3 \equiv \beta_3 - \beta_1$. Note that since t is in the interior of $\triangle v_1 v_2 v_3$, it holds that $b_i \in (0, \pi)$ for $i = 1, 2, 3$. Note that $b_1 + b_2 + b_3 = 2\pi$.

For each $i = 1, 2, 3$, let α_i denote the angle of the direction $\overrightarrow{su_i}$. According to our definition of pivot vertices, $u_i = u_j$ if and only if $\alpha_i = \alpha_j$ for any two $i, j \in \{1, 2, 3\}$. We define three angles a_i for $i = 1, 2, 3$ as follows (e.g., see Fig. 2). Define a_1 as the smallest angle we need to rotate the direction $\overrightarrow{su_1}$ counterclockwise to $\overrightarrow{su_2}$; define a_2 as the smallest angle we need to rotate the direction $\overrightarrow{su_2}$ clockwise to $\overrightarrow{su_3}$; define a_3 as the smallest angle we need to rotate the direction $\overrightarrow{su_3}$ clockwise to $\overrightarrow{su_1}$. Hence, $a_1 \equiv \alpha_2 - \alpha_1$, $a_2 \equiv \alpha_3 - \alpha_2$, and $a_3 \equiv \alpha_1 - \alpha_3$.

We refer to the case where $a_i = b_i$ for each $i = 1, 2, 3$ as the *special case*.

Lemma 11. *The π -range $R_\pi(s, t)$ is determined as follows (e.g., see Fig. 13).*

$$R_\pi(s, t) = \begin{cases} (\alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}), \alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi) & \text{if } \delta > 0, \\ (\alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}) - \pi, \alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta})) & \text{if } \delta < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 > \delta_2, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 < \delta_2, \\ \emptyset & \text{if } \delta = 0 \text{ and } \delta_1 = \delta_2, \end{cases}$$

where $\delta = \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)}$, $\delta_1 = \frac{\cos(\beta_2 - \beta_1) - \cos(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)}$, and $\delta_2 = \frac{\cos(\beta_3 - \beta_1) - \cos(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)}$. Further, $a_i = b_i$ for each $i = 1, 2, 3$ (i.e., the special case) if and only if $\delta = 0$ and $\delta_1 = \delta_2$.

We defer the proof of Lemma 11 to Section 4.5. According to Lemma 11, if the special case happens, then $R_\pi(s, t)$ is empty; otherwise, it is an open range of size exactly π .

Now we are back to our original problem to determine the range $R(s, t)$ for a non-degenerate farthest point $t \in \mathcal{I}$ of s . Since there are exactly three shortest s - t paths $\pi_i(s, t)$ for $i = 1, 2, 3$, the three paths must be canonical. To see this, by Observation 1, t is in the interior of $\triangle v_1 v_2 v_3$. Further,

it is easy to see that no two of the three paths cross each other since otherwise there would be more than three shortest s - t paths, this implies that the second property of the canonical paths holds. Let $R_\pi(s, t)$ be the π -range of s with respect to t and the above three shortest paths. We have the following result.

Lemma 12. $R(s, t) = R_\pi(s, t) \cap R_f(s)$.

Proof: For each $i = 1, 2, 3$, let u_i and v_i be the s -pivot and t -pivot of the shortest path $\pi_i(s, t)$, respectively. Since s and t have only three shortest paths, $U_t(s) = \{v_1, v_2, v_3\}$ and $U_s(t) = \{u_1, u_2, u_3\}$. Further, for each $i = 1, 2, 3$, v_i have only one s -pivot, which is u_i .

1. Consider any $r_s \in R(s, t)$. Clearly, $r_s \in R_f(s)$. By Lemma 4(1), there is a direction r_t for t with a speed $\tau \geq 0$ such that when s moves along r_s with unit speed and t moves along r_t with speed τ , each $v \in U_t(s)$ has a coupled s -pivot u with $d'_{u,v}(s, t) < 0$. Since $U_t(s) = \{v_1, v_2, v_3\}$ and each v_i has only one s -pivot u_i for each $1 \leq i \leq 3$, we have $d'_{u_i, v_i}(s, t) < 0$ for $i = 1, 2, 3$. Hence, r_s is in $R_\pi(s, t)$.
2. Consider any r_s in $R_\pi(s, t) \cap R_f(s)$. First of all, r_s is a free direction. Since r_s is in $R_\pi(s, t)$, there exists a direction r_t for t with a speed $\tau \geq 0$ such that if s moves along r_s for unit speed and t moves along r_t with speed τ , then $d'_{u_i, v_i}(s, t) < 0$ for $i = 1, 2, 3$. Since $U_t(s) = \{v_1, v_2, v_3\}$, according to Lemma 4(1), r_s is in $R(s, t)$.

The lemma thus follows. \square

4.5 Proof of Lemma 11

Consider any direction r_s for moving s and r_t for moving t . Let θ_s denote the angle of the direction r_s . Let θ_t denote the angle of r_t . According to our analysis for Equation (1), we can obtain the derivatives of the three functions $d'_{u_i, v_i}(s, t)$ for $i = 1, 2, 3$, as follows.

$$\begin{cases} d'_{u_1, v_1}(s, t) = -\cos(\alpha_1 - \theta_s) - \tau \cdot \cos(\beta_1 - \theta_t), \\ d'_{u_2, v_2}(s, t) = -\cos(\alpha_2 - \theta_s) - \tau \cdot \cos(\beta_2 - \theta_t), \\ d'_{u_3, v_3}(s, t) = -\cos(\alpha_3 - \theta_s) - \tau \cdot \cos(\beta_3 - \theta_t). \end{cases} \quad (5)$$

Therefore, for each $i = 1, 2, 3$, $d'_{u_i, v_i}(s, t)$ is a function of θ_s , θ_t , and τ . In order to simplify our proof for Lemma 11, we first give the following lemma.

Lemma 13. *A direction r_s is in $R_\pi(s, t)$ if and only if there exist $\theta_t \in [0, 2\pi)$ and $\tau \geq 0$ such that $d'_{u_1, v_1}(s, t) = 0$ and $d'_{u_i, v_i}(s, t) < 0$ for $i = 2, 3$ (the same result holds if we switch the index 1 with 2 or 3).*

Proof: In order to make the notation consistent with the rest of Section 4.5, instead of proving the statement of the lemma, we prove the following statement (which is essentially the same as the lemma): r_s is in $R_\pi(s, t)$ if and only if there exist θ_t and $\tau \geq 0$ such that $d'_{u_2, v_2}(s, t) = 0$ and $d'_{u_i, v_i}(s, t) < 0$ for $i = 1, 3$.

Let $x = \alpha_1 - \theta_s$ and $y = \beta_1 - \theta_t$. Hence, we have the following:

$$\begin{cases} d'_{u_1, v_1}(s, t) = -\cos(x) - \tau \cdot \cos(y), \\ d'_{u_2, v_2}(s, t) = -\cos(\alpha_2 - \alpha_1 + x) - \tau \cdot \cos(\beta_2 - \beta_1 + y), \\ d'_{u_3, v_3}(s, t) = -\cos(\alpha_3 - \alpha_1 + x) - \tau \cdot \cos(\beta_3 - \beta_1 + y). \end{cases}$$

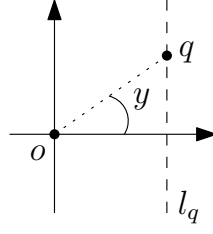


Fig. 14. Illustrating the polar coordinate of q with $\tau = |oq|$.

Consider any fixed direction r_s . The angle θ_s is also fixed, and thus x is fixed. Hence, for each $i = 1, 2, 3$, $d'_{u_i, v_i}(s, t)$ is a function of θ_t and τ , and thus also a function of y and τ . In the following proof, we will use $d'_i(y, \tau)$ to represent $d'_{u_i, v_i}(s, t)$ for each $i = 1, 2, 3$.

We first prove one direction of the lemma. Suppose r_s is in $R_\pi(s, t)$. Then there exist y and $\tau \geq 0$ such that $d'_i(y, \tau) < 0$ for each $i = 1, 2, 3$. Our goal is to prove that there exist y' and $\tau' \geq 0$ such that $d'_2(y', \tau') = 0$ and $d'_i(y', \tau') < 0$ for $i = 1, 3$.

The idea is to change y and τ simultaneously in such a way that $d'_2(y, \tau)$ is constant, but $d'_1(y, \tau)$ strictly increases while $d'_3(y, \tau)$ strictly decreases, and we keep making the change until $d'_1(y, \tau)$ becomes zero. To this end, we will find a way that when we change τ and y simultaneously, $\tau \cdot \cos(y)$ is constant, $-\tau \cdot \cos(\beta_2 - \beta_1 + y)$ increases, and $-\tau \cdot \cos(\beta_3 - \beta_1 + y)$ decreases, as follows.

Since $\cos(\beta_2 - \beta_1 + y) = \cos(\beta_2 - \beta_1) \cdot \cos(y) - \sin(\beta_2 - \beta_1) \cdot \sin(y)$, we have

$$d'_2(y, \tau) = -\cos(\alpha_2 - \alpha_1 + x) - \tau \cdot \cos(y) \cdot \cos(\beta_2 - \beta_1) + \tau \cdot \sin(y) \cdot \sin(\beta_2 - \beta_1).$$

Similarly, we derive

$$d'_3(y, \tau) = -\cos(\alpha_3 - \alpha_1 + x) - \tau \cdot \cos(y) \cdot \cos(\beta_3 - \beta_1) + \tau \cdot \sin(y) \cdot \sin(\beta_3 - \beta_1).$$

Consider a point q in a polar coordinate system with coordinate (τ, y) , i.e., $\tau = |oq|$ and the polar angle of q is y , where o is the origin (e.g., see Fig. 14). Let l_q be the vertical line through q . Note that if $\tau = 0$, then $q = o$ and l_q is the vertical line through o .

For any point p on l_q , let $\tau_p = |op|$ and let y_p be the polar angle of p . Suppose we move p on l_q from top to bottom, it is easy to see that $\tau_p \cos(y_p)$ is constant and $\tau_p \sin(y_p)$ is strictly decreasing. Based on this observation, we will find y' and $\tau' \geq 0$ such that $d'_2(y', \tau') = 0$ and $d'_i(y', \tau') < 0$ for $i = 1, 3$, as follows.

Recall that $\beta_1 - \beta_2 \equiv b_1$ and $b_1 \in (0, \pi)$. Thus, $\sin(\beta_2 - \beta_1) < 0$. Similarly, since $\beta_3 - \beta_1 \equiv b_3$ and $b_3 \in (0, \pi)$, $\sin(\beta_3 - \beta_1) > 0$. Hence, if we change the values of y and τ simultaneously such that q moves along l_q downwards, then $d'_1(y, \tau)$ does not change, but $d'_2(y, \tau)$ strictly increases while $d'_3(y, \tau)$ strictly decreases. We keep making the above change until at some moment $d'_2(y, \tau)$ becomes zero, at which moment we have $d'_2(y, \tau) = 0$ and $d'_i(y, \tau) < 0$ for $i = 1, 3$.

The above proves one direction of the lemma. Next, we prove the other direction.

Suppose there exist y and $\tau \geq 0$ such that $d'_2(y, \tau) = 0$ and $d'_i(y, \tau) < 0$ for $i = 1, 3$. Our goal is to show that there exist y' and $\tau' \geq 0$ such that $d'_i(y', \tau') < 0$ for $i = 1, 2, 3$.

1. If $\tau > 0$, depending on whether $\cos(\beta_2 - \beta_1 + y) = 0$, there are two subcases.
 - (a) If $\cos(\beta_2 - \beta_1 + y) = 0$, then it is always possible to change y infinitesimally such that $\cos(\beta_2 - \beta_1 + y) > 0$. Due to $\tau > 0$, $d'_2(y, \tau) < 0$. Also, since the change of y is infinitesimal and both $\cos(y)$ and $\cos(\beta_3 - \beta_1 + y)$ are continuous functions, we still have $d'_i(y, \tau) < 0$ for $i = 2, 3$. We are done with the proof.
 - (b) If $\cos(\beta_2 - \beta_1 + y) \neq 0$, depending on whether $\cos(\beta_2 - \beta_1 + y)$ is positive or negative, we can infinitesimally increase or decrease τ , such that τ is still positive and $d'_i(y, \tau) < 0$ holds for each $i = 1, 2, 3$. We are done with the proof.

2. If $\tau = 0$, then again depending on whether $\cos(\beta_2 - \beta_1 + y)$ is positive, negative, or zero, there are three subcases.
- (a) If $\cos(\beta_2 - \beta_1 + y) > 0$, then we can increase τ infinitesimally such that $\tau > 0$ and $d'_i(y, \tau) < 0$ for $i = 1, 2, 3$.
 - (b) If $\cos(\beta_2 - \beta_1 + y) < 0$, then we first change y to $y + \pi$ (i.e., the moving direction of t is reversed). After this, since $\tau = 0$, we still have $d'_2(y, \tau) = 0$ and $d'_i(y, \tau) < 0$ for $i = 1, 3$. However, the difference is that now $\cos(\beta_2 - \beta_1 + y) > 0$ for the new y . Then, we can use the same analysis as the first subcase.
 - (c) If $\cos(\beta_2 - \beta_1 + y) = 0$, then we first slightly change y such that $\cos(\beta_2 - \beta_1 + y) > 0$. Since $\tau = 0$, we still have $d'_2(y, \tau) = 0$ and $d'_i(y, \tau) < 0$ for $i = 1, 3$. However, the difference is that now $\cos(\beta_2 - \beta_1 + y) > 0$ for the new y . Then, we can use the same analysis as the first subcase.

This completes the proof of the lemma. \square

To simplify the notation, let $w_i = -d'_{u_i, v_i}(s, t)$ for each $i = 1, 2, 3$. Let $x = \alpha_1 - \theta_s$ and $y = \beta_1 - \theta_t$. Then, by Equation (5), we have the following.

$$\begin{cases} w_1 = \cos(x) + \tau \cdot \cos(y), \\ w_2 = \cos(\alpha_2 - \alpha_1 + x) + \tau \cdot \cos(\beta_2 - \beta_1 + y), \\ w_3 = \cos(\alpha_3 - \alpha_1 + x) + \tau \cdot \cos(\beta_3 - \beta_1 + y). \end{cases} \quad (6)$$

Once r_s is fixed, both θ_s and x are fixed. Also, given any θ_t , we can determine y , and vice versa. In the following, for each $i = 1, 2, 3$, we consider w_i implicitly as a function of $y \in [0, 2\pi)$ and $\tau \geq 0$. By Lemma 13, $R_\pi(s, t)$ consists of all directions r_s for s such that there exist y and $\tau \geq 0$ with $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$.

Let $w_1 = 0$. Then $\cos(x) + \tau \cdot \cos(y) = 0$. Depending on whether $\cos(x)$ is positive, negative, or zero, there are three cases.

The case $\cos(x) > 0$. In this case, since $\tau \cdot \cos(y) = -\cos(x)$, we have $\tau \cdot \cos(y) < 0$ and $\tau \neq 0$. Since $\tau \geq 0$, we obtain $\tau > 0$ and $\cos(y) < 0$, implying that $y \in (\pi/2, 3\pi/2)$. Further, we have $\tau = -\cos(x)/\cos(y)$. By replacing τ with $-\cos(x)/\cos(y)$ in Equation (6) for w_2 , we obtain

$$\begin{aligned} w_2 &= \cos(\alpha_2 - \alpha_1) \cdot \cos(x) - \sin(\alpha_2 - \alpha_1) \cdot \sin(x) \\ &\quad - \frac{\cos(x)}{\cos(y)} \cdot [\cos(\beta_2 - \beta_1) \cdot \cos(y) - \sin(\beta_2 - \beta_1) \cdot \sin(y)] \\ &= \cos(\alpha_2 - \alpha_1) \cdot \cos(x) - \sin(\alpha_2 - \alpha_1) \cdot \sin(x) \\ &\quad - \cos(\beta_2 - \beta_1) \cdot \cos(x) + \sin(\beta_2 - \beta_1) \cdot \tan(y) \cdot \cos(x). \end{aligned}$$

Since $\cos(x) > 0$, if we divide the right side of the above formula by $\cos(x)$, we obtain that $w_2 > 0$ if and only if

$$\cos(\alpha_2 - \alpha_1) - \sin(\alpha_2 - \alpha_1) \cdot \tan(x) - \cos(\beta_2 - \beta_1) + \sin(\beta_2 - \beta_1) \cdot \tan(y) > 0.$$

Recall that $\beta_1 - \beta_2 \equiv b_1$ and $b_1 \in (0, \pi)$, and thus $\sin(\beta_2 - \beta_1) < 0$. Hence, the above inequality is equivalent to

$$\tan(y) < \frac{\cos(\beta_2 - \beta_1) - \cos(\alpha_2 - \alpha_1) + \sin(\alpha_2 - \alpha_1) \cdot \tan(x)}{\sin(\beta_2 - \beta_1)}. \quad (7)$$

Therefore, we obtained that $w_2 > 0$ if and only if Inequality (7) holds.

Similarly, for w_3 , we can obtain that $w_3 > 0$ if and only if the following inequality holds:

$$\tan(y) > \frac{\cos(\beta_3 - \beta_1) - \cos(\alpha_3 - \alpha_1) + \sin(\alpha_3 - \alpha_1) \cdot \tan(x)}{\sin(\beta_3 - \beta_1)}. \quad (8)$$

Note that to obtain Inequality (8), we need to use the fact that $\sin(\beta_3 - \beta_1) > 0$, which is due to $\beta_3 - \beta_1 \equiv b_3$ and $b_3 \in (0, \pi)$.

As a summary, the above shows that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if both Inequalities (7) and (8) hold. Recall that since $\cos(x) > 0$, y is in $(\pi/2, 3\pi/2)$. Further, note that we can find an angle $y \in (\pi/2, 3\pi/2)$ such that both (7) and (8) hold if and only if the right side of Inequality (7) is strictly larger than the right side of Inequality (8). Hence, we obtain the following observation.

Observation 4 *For the case $\cos(x) > 0$, there exist y and $\tau \geq 0$ such that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if the right side of Inequality (7) is strictly larger than the right side of Inequality (8), which is equivalent to the following inequality:*

$$\delta \cdot \tan(x) < \delta_1 - \delta_2, \quad (9)$$

where δ , δ_1 , and δ_2 are defined as in the statement of Lemma 11.

Proof: Recall that

$$\delta = \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)},$$

$$\delta_1 = \frac{\cos(\beta_2 - \beta_1) - \cos(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)}, \text{ and } \delta_2 = \frac{\cos(\beta_3 - \beta_1) - \cos(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)}.$$

That the right side of Inequality (7) is larger than the right side of Inequality (8) is equivalent to the following

$$\frac{\sin(\alpha_3 - \alpha_1) \cdot \tan(x)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1) \cdot \tan(x)}{\sin(\beta_2 - \beta_1)} < \frac{\cos(\beta_2 - \beta_1) - \cos(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)} - \frac{\cos(\beta_3 - \beta_1) - \cos(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)},$$

which is further equivalent to Inequality (9). \square

Recall that $\cos(x) > 0$. Thus, $x \in (-\pi/2, \pi/2)$.

If $\delta > 0$, Inequality (9) is equivalent to $\tan(x) < \frac{\delta_1 - \delta_2}{\delta}$. Consequently, since $x \in (-\pi/2, \pi/2)$, we obtain that $x \in (-\pi/2, \arctan(\frac{\delta_1 - \delta_2}{\delta}))$.

If $\delta < 0$, Inequality (9) is equivalent to $\tan(x) > \frac{\delta_1 - \delta_2}{\delta}$, implying that $x \in (\arctan(\frac{\delta_1 - \delta_2}{\delta}), \pi/2)$.

If $\delta = 0$, Inequality (9) is equivalent to $0 < \delta_1 - \delta_2$. Hence, if $\delta_1 > \delta_2$, then any x can make the inequality hold, implying that $x \in (-\pi/2, \pi/2)$; otherwise, no x can make it hold.

In summary, for the case $\cos(x) > 0$, there exist y and $\tau \geq 0$ such that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if: $x \in (-\pi/2, \arctan(\frac{\delta_1 - \delta_2}{\delta}))$ when $\delta > 0$; $x \in (\arctan(\frac{\delta_1 - \delta_2}{\delta}), \pi/2)$ when $\delta < 0$; $x \in (-\pi/2, \pi/2)$ when $\delta = 0$ and $\delta_1 > \delta_2$.

The case $\cos(x) < 0$. In this case, $x \in (\pi/2, 3\pi/2)$. The analysis is similar to the above case. We omit the details and only give the result below.

There exist y and $\tau \geq 0$ such that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if: $x \in (\arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi, 3\pi/2)$ when $\delta > 0$; $x \in (\pi/2, \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi)$ when $\delta < 0$; $x \in (\pi/2, 3\pi/2)$ when $\delta = 0$ and $\delta_1 < \delta_2$.

The case $\cos(x) = 0$. In this case $x = \pm\pi/2$. The analysis for this case is different from the above two cases. We will show the following result: There exist y and $\tau \geq 0$ such that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if: $x = -\pi/2$ when $\delta > 0$; $x = \pi/2$ when $\delta < 0$ (there is no value for x when $\delta = 0$).

By Lemma 13, there exist y and $\tau \geq 0$ such that $w_1 = 0$ and $w_i > 0$ for $i = 2, 3$ if and only if there exist y and $\tau \geq 0$ such that $w_i > 0$ for $i = 1, 2, 3$. For convenience, in this case, we will find all directions r_s such that there exist y and $\tau \geq 0$ with $w_i > 0$ for $i = 1, 2, 3$.

Since $\cos(x) = 0$, we have $w_1 = \tau \cdot \cos(y)$. Since we require $w_1 > 0$, τ cannot be 0 and thus $\tau > 0$ must hold. Therefore, we have $\cos(y) > 0$ and $y \in (-\pi/2, \pi/2)$. In addition, $\tau = \frac{w_1}{\cos(y)}$. By replacing τ with $\frac{w_1}{\cos(y)}$ and setting $\cos(x)$ to 0 in Equation (6) for w_2 , we obtain the following

$$\begin{aligned} w_2 &= \cos(\alpha_2 - \alpha_1) \cdot \cos(x) - \sin(\alpha_2 - \alpha_1) \cdot \sin(x) \\ &\quad + \frac{w_1}{\cos(y)} \cdot [\cos(\beta_2 - \beta_1) \cdot \cos(y) - \sin(\beta_2 - \beta_1) \cdot \sin(y)] \\ &= -\sin(\alpha_2 - \alpha_1) \cdot \sin(x) + \cos(\beta_2 - \beta_1) \cdot w_1 - \sin(\beta_2 - \beta_1) \cdot \tan(y) \cdot w_1. \end{aligned}$$

Hence, $w_2 > 0$ if and only if the following holds

$$\sin(\beta_2 - \beta_1) \cdot \tan(y) \cdot w_1 < -\sin(\alpha_2 - \alpha_1) \cdot \sin(x) + \cos(\beta_2 - \beta_1) \cdot w_1.$$

Recall that since $\beta_1 - \beta_2 = b_1$ and $b_1 \in (0, \pi)$, $\sin(\beta_2 - \beta_1) < 0$. Since $w_1 > 0$, the above inequality is equivalent to the following

$$\tan(y) > \frac{-\sin(\alpha_2 - \alpha_1) \cdot \sin(x) + \cos(\beta_2 - \beta_1) \cdot w_1}{\sin(\beta_2 - \beta_1) \cdot w_1} \quad (10)$$

For w_3 , by the similar analysis (we also need to use the fact that $\sin(\beta_3 - \beta_1) > 0$), $w_3 > 0$ if and only if the following holds

$$\tan(y) < \frac{-\sin(\alpha_3 - \alpha_1) \cdot \sin(x) + \cos(\beta_3 - \beta_1) \cdot w_1}{\sin(\beta_3 - \beta_1) \cdot w_1} \quad (11)$$

Based on the above discussion, $w_i > 0$ for $i = 1, 2, 3$ if and only if we can find $y \in (-\pi/2, \pi/2)$ such that both Inequalities (10) and (11) hold. It is not difficult to see that we can find $y \in (-\pi/2, \pi/2)$ such that both Inequalities (10) and (11) hold if and only if the right side of Inequality (10) is strictly less than the right side of Inequality (11), i.e.,

$$\frac{-\sin(\alpha_2 - \alpha_1) \cdot \sin(x) + \cos(\beta_2 - \beta_1) \cdot w_1}{\sin(\beta_2 - \beta_1) \cdot w_1} < \frac{-\sin(\alpha_3 - \alpha_1) \cdot \sin(x) + \cos(\beta_3 - \beta_1) \cdot w_1}{\sin(\beta_3 - \beta_1) \cdot w_1}. \quad (12)$$

Since $w_1 > 0$, we have the following observation.

Observation 5 $w_i > 0$ for $i = 1, 2, 3$ if and only if there exists $w_1 > 0$ such that

$$\delta \cdot \sin(x) < w_1 \cdot [\cot(\beta_3 - \beta_1) - \cot(\beta_2 - \beta_1)].$$

Proof: The left side of Inequality (12) is equal to $\cot(\beta_2 - \beta_1) - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)} \cdot \frac{\sin(x)}{w_1}$ and the right side is equal to $\cot(\beta_3 - \beta_1) - \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} \cdot \frac{\sin(x)}{w_1}$. Hence, Inequality (12) is equivalent to the following

$$\frac{\sin(x)}{w_1} \cdot \left[\frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)} \right] < \cot(\beta_3 - \beta_1) - \cot(\beta_2 - \beta_1).$$

Recall that $\delta = \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)}$. Since $w_1 > 0$, the observation is obtained. \square

We also have the following observation.

Lemma 14. *It holds that $\cot(\beta_3 - \beta_1) - \cot(\beta_2 - \beta_1) < 0$.*

Proof: Note that $\beta_3 - \beta_1 \equiv b_3$ and $\beta_2 - \beta_1 \equiv b_3 + b_2$. Hence, to prove the lemma, it is sufficient to prove $\cot(b_3) < \cot(b_3 + b_2)$.

Recall that b_i is in $(0, \pi)$ for each $i = 1, 2, 3$. Since $b_1 + b_2 + b_3 = 2\pi$, it holds that $b_3 + b_2 \in (\pi, 2\pi)$.

Note that $\cot(b_3) = \cot(b_3 + \pi)$. Due to $b_3 \in (0, \pi)$, $b_3 + \pi \in (\pi, 2\pi)$. Hence, both $b_3 + b_2$ and $b_3 + \pi$ are in $(\pi, 2\pi)$. Since $b_2 \in (0, \pi)$, $b_3 + b_2 < b_3 + \pi$. Because $\cot(\cdot)$ is a strictly decreasing function on $(\pi, 2\pi)$, we obtain $\cot(b_3 + b_2) > \cot(b_3 + \pi) = \cot(b_3)$, which proves the lemma. \square

Recall that x is either $\pi/2$ or $-\pi/2$.

In light of Lemma 14, if $\delta > 0$, then there exists $w_1 > 0$ such that the inequality in Observation 5 is satisfied if and only if $\sin(x) < 0$, i.e., $x = -\pi/2$. Therefore, if $\delta > 0$, by Observation 5, $w_i > 0$ for $i = 1, 2, 3$ if and only if $x = -\pi/2$.

Similarly, we can obtain that if $\delta < 0$, then $w_i > 0$ for $i = 1, 2, 3$ if and only if $x = \pi/2$.

If $\delta = 0$, then for any x and any $w_1 > 0$, the inequality in Observation 5 can never be satisfied. Therefore, for any x , it is not possible to have $w_i < 0$ for all $i = 1, 2, 3$.

A summary of all three cases. Let $R(x)$ be the set of values for x such that there exist y and $\tau \geq 0$ with $w_i > 0$ for $i = 1, 2, 3$. In other words, since $x = \alpha_1 - \theta_t$, $R(s) = \{\alpha_1 - \theta_t \mid \theta_t \in R_\pi(s, t)\}$. By our discussions for all three cases $\cos(x) > 0$, $\cos(x) < 0$, and $\cos(x) = 0$, we can obtain the following:

If $\delta > 0$, we have

$$R(x) = \begin{cases} (-\pi/2, \arctan(\frac{\delta_1 - \delta_2}{\delta})) & \text{for } \cos(x) > 0, \\ (\arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi, 3\pi/2) = (\arctan(\frac{\delta_1 - \delta_2}{\delta}) - \pi, -\pi/2) & \text{for } \cos(x) < 0, \\ \{-\pi/2\} & \text{for } \cos(x) = 0. \end{cases}$$

Therefore, if $\delta > 0$, $R(x)$ is the union of the above three intervals, which is $(\arctan(\frac{\delta_1 - \delta_2}{\delta}) - \pi, \arctan(\frac{\delta_1 - \delta_2}{\delta}))$.

Similarly, if $\delta < 0$, we have

$$R(x) = \begin{cases} (\arctan(\frac{\delta_1 - \delta_2}{\delta}), \pi/2) & \text{for } \cos(x) > 0, \\ (\pi/2, \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi) & \text{for } \cos(x) < 0, \\ \{\pi/2\} & \text{for } \cos(x) = 0. \end{cases}$$

Therefore, if $\delta < 0$, $R(x) = (\arctan(\frac{\delta_1 - \delta_2}{\delta}), \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi)$.

Finally, if $\delta = 0$, depending on the values of δ_1 and δ_2 , there are three cases. If $\delta_1 > \delta_2$, $R(x) = (-\pi/2, \pi/2)$. If $\delta_1 < \delta_2$, $R(x) = (\pi/2, 3\pi/2)$. If $\delta_1 = \delta_2$, $R(x) = \emptyset$.

As a summary, we obtain the following

$$R(x) = \begin{cases} (\arctan(\frac{\delta_1 - \delta_2}{\delta}) - \pi, \arctan(\frac{\delta_1 - \delta_2}{\delta})) & \text{if } \delta > 0, \\ (\arctan(\frac{\delta_1 - \delta_2}{\delta}), \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi) & \text{if } \delta < 0, \\ (-\pi/2, \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 > \delta_2, \\ (\pi/2, 3\pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 < \delta_2, \\ \emptyset & \text{if } \delta = 0 \text{ and } \delta_1 = \delta_2, \end{cases}$$

Since $R(s) = \{\alpha_1 - \theta_t \mid \theta_t \in R_\pi(s, t)\}$, we obtain the range $R_\pi(s, t)$ for θ_s as follows.

$$R_\pi(s, t) = \begin{cases} (\alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}), \alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}) + \pi) & \text{if } \delta > 0, \\ (\alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta}) - \pi, \alpha_1 - \arctan(\frac{\delta_1 - \delta_2}{\delta})) & \text{if } \delta < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 > \delta_2, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 < \delta_2, \\ \emptyset & \text{if } \delta = 0 \text{ and } \delta_1 = \delta_2, \end{cases}$$

To prove Lemma 11, it remains to show that $\delta = 0$ and $\delta_1 = \delta_2$ if and only if $a_i = b_i$ for each $i = 1, 2, 3$. To this end, we first give the following lemma.

Lemma 15. *If $\delta = 0$ and $\delta_1 = \delta_2$, then $\delta_1 = \delta_2 = 0$.*

Proof: Recall that

$$\delta = \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)},$$

$$\delta_1 = \frac{\cos(\beta_2 - \beta_1) - \cos(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)}, \text{ and } \delta_2 = \frac{\cos(\beta_3 - \beta_1) - \cos(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)}.$$

For the purpose of differentiation, we use $\delta(1, 2, 3)$, $\delta_1(1, 2, 3)$, and $\delta_2(1, 2, 3)$ to represent the above δ , δ_1 , and δ_2 , respectively. We use $(1, 2, 3)$ because our previous discussion considered $d'_{u_i, v_i}(s, t)$ in the order of 1, 2, 3.

Suppose $\delta(1, 2, 3) = 0$ and $\delta_1(1, 2, 3) = \delta_2(1, 2, 3)$. In the following, we give an “indirect” approach to prove $\delta_1(1, 2, 3) = \delta_2(1, 2, 3) = 0$.

Our previous discussion has proved that $R_\pi(s, t) = \emptyset$ if and only if $\delta(1, 2, 3) = 0$ and $\delta_1(1, 2, 3) = \delta_2(1, 2, 3)$. In other words, there is no value for θ_s such that there exist θ_t and $\tau \geq 0$ with $w_i > 0$ for $i = 1, 2, 3$ if and only if $\delta(1, 2, 3) = 0$ and $\delta_1(1, 2, 3) = \delta_2(1, 2, 3)$.

Our previous analysis considered the three functions w_i in the order of 1, 2, 3. If we consider them in the order of 2, 3, 1, by using the same analysis, we can obtain that $R_\pi(s, t) = \emptyset$ if and only if $\delta(2, 3, 1) = 0$ and $\delta_1(2, 3, 1) = \delta_2(2, 3, 1)$, where

$$\delta(2, 3, 1) = \frac{\sin(\alpha_1 - \alpha_2)}{\sin(\beta_1 - \beta_2)} - \frac{\sin(\alpha_3 - \alpha_2)}{\sin(\beta_3 - \beta_2)},$$

$$\delta_1(2, 3, 1) = \frac{\cos(\beta_3 - \beta_2) - \cos(\alpha_3 - \alpha_2)}{\sin(\beta_3 - \beta_2)}, \text{ and } \delta_2(2, 3, 1) = \frac{\cos(\beta_1 - \beta_2) - \cos(\alpha_1 - \alpha_2)}{\sin(\beta_1 - \beta_2)}.$$

Another way to think about this is that we can replace 1, 2, 3 by 2, 3, 1, respectively, in $\delta(1, 2, 3)$, $\delta_1(1, 2, 3)$, and $\delta_2(1, 2, 3)$. The main reason why the same analysis as before still works is that if we follow the order of 2, 3, 1, the vertices of u_2, u_3, u_1 are still in the counterclockwise order around s and the vertices of v_2, v_3, v_1 are still in the clockwise order around t . We omit the detailed analysis.

Similarly, if we consider the three functions w_i in the order of 3, 1, 2, by using the same analysis, we can obtain that $R_\pi(s, t) = \emptyset$ if and only if $\delta(3, 1, 2) = 0$ and $\delta_2(3, 1, 2) = \delta_3(3, 1, 2)$, where

$$\delta(3, 1, 2) = \frac{\sin(\alpha_2 - \alpha_3)}{\sin(\beta_2 - \beta_3)} - \frac{\sin(\alpha_1 - \alpha_3)}{\sin(\beta_1 - \beta_3)},$$

$$\delta_1(3, 1, 2) = \frac{\cos(\beta_1 - \beta_3) - \cos(\alpha_1 - \alpha_3)}{\sin(\beta_1 - \beta_3)}, \text{ and } \delta_2(3, 1, 2) = \frac{\cos(\beta_2 - \beta_3) - \cos(\alpha_2 - \alpha_3)}{\sin(\beta_2 - \beta_3)}.$$

Recall that $\delta(1, 2, 3) = 0$ and $\delta_1(1, 2, 3) = \delta_2(1, 2, 3)$. Thus, $R_\pi(s, t) = \emptyset$. According to the above discussion, $\delta(2, 3, 1) = 0$ and $\delta_1(2, 3, 1) = \delta_2(2, 3, 1)$, and $\delta(3, 1, 2) = 0$ and $\delta_2(3, 1, 2) = \delta_3(3, 1, 2)$. In the following, we show that $\delta_1(1, 2, 3) = \delta_2(1, 2, 3) = \delta_1(2, 3, 1) = \delta_2(2, 3, 1) = \delta_2(3, 1, 2) = \delta_3(3, 1, 2) = 0$, which will prove the lemma.

First of all, $\delta_2(1, 2, 3) = -\delta_1(3, 1, 2)$ holds because $\cos(\beta_3 - \beta_1) = \cos(\beta_1 - \beta_3)$, $\cos(\alpha_3 - \alpha_1) = \cos(\alpha_1 - \alpha_3)$, and $\sin(\beta_3 - \beta_1) = -\sin(\beta_1 - \beta_3)$.

Similarly, we can obtain $\delta_1(2, 3, 1) = -\delta_2(3, 1, 2)$ and $\delta_1(1, 2, 3) = -\delta_2(2, 3, 1)$.

Due to $\delta_2(1, 2, 3) = -\delta_1(3, 1, 2)$ and $\delta_1(2, 3, 1) = -\delta_2(3, 1, 2)$, we derive $\delta_1(1, 2, 3) = \delta_2(1, 2, 3) = -\delta_1(3, 1, 2) = -\delta_2(3, 1, 2) = \delta_1(2, 3, 1) = \delta_2(2, 3, 1)$. In particular, we have $\delta_1(1, 2, 3) = \delta_2(2, 3, 1)$.

The above has obtained that both $\delta_1(1, 2, 3) = -\delta_2(2, 3, 1)$ and $\delta_1(1, 2, 3) = \delta_2(2, 3, 1)$ hold, implying that $\delta_1(1, 2, 3) = \delta_2(2, 3, 1) = 0$. Consequently, $\delta_1(1, 2, 3) = \delta_2(1, 2, 3) = \delta_1(2, 3, 1) = \delta_2(2, 3, 1) = \delta_2(3, 1, 2) = \delta_3(3, 1, 2) = 0$.

The lemma thus follows. \square

Lemma 16. $\delta = 0$ and $\delta_1 = \delta_2$ if and only if $a_i = b_i$ for each $i = 1, 2, 3$.

Proof: Recall that $a_1 \equiv \alpha_2 - \alpha_1$, $a_2 \equiv \alpha_3 - \alpha_2$, $a_3 \equiv \alpha_1 - \alpha_3$, $b_1 \equiv \beta_1 - \beta_2$, $b_2 \equiv \beta_2 - \beta_3$, and $b_3 \equiv \beta_3 - \beta_1$. We first prove one direction of the lemma. Suppose $a_i = b_i$ for each $i = 1, 2, 3$. Then,

$$\delta = \frac{\sin(\alpha_3 - \alpha_1)}{\sin(\beta_3 - \beta_1)} - \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\beta_2 - \beta_1)} = \frac{-\sin(a_3)}{\sin(b_3)} - \frac{\sin(a_1)}{-\sin(b_1)} = 0.$$

Also, we have $\cos(\beta_2 - \beta_1) = \cos(b_1)$ and $\cos(\alpha_2 - \alpha_1) = \cos(a_1)$. Since $a_1 = b_1$, we obtain $\delta_1 = 0$. Similarly, we have $\cos(\beta_3 - \beta_1) = \cos(b_3)$ and $\cos(\alpha_3 - \alpha_1) = \cos(a_3)$. Since $a_3 = b_3$, we obtain $\delta_2 = 0$. This proves that $\delta = 0$ and $\delta_1 = \delta_2$.

Next we prove the other direction of the lemma. Suppose $\delta = 0$ and $\delta_1 = \delta_2$. In the following we prove $a_i = b_i$ for $i = 1, 2, 3$. By Lemma 15, we know that $\delta_1 = \delta_2 = 0$.

Since $\delta_1 = 0$, it holds that $\cos(\beta_2 - \beta_1) = \cos(\alpha_2 - \alpha_1)$, i.e., $\cos(b_1) = \cos(a_1)$.

Similarly, since $\delta_2 = 0$, it holds that $\cos(\beta_3 - \beta_1) = \cos(\alpha_3 - \alpha_1)$, i.e., $\cos(b_3) = \cos(a_3)$.

On the other hand, $\delta = 0$ implies that $\frac{\sin(a_3)}{\sin(b_3)} = \frac{\sin(a_1)}{\sin(b_1)}$.

Note that $b_1 + b_2 + b_3 = 2\pi$. Recall that two or even all three of u_1, u_2, u_3 may be the same. According to our definitions of the three angles a_1, a_2, a_3 , if $u_1 = u_2 = u_3$, then $a_1 = a_2 = a_3 = 0$; otherwise, $a_1 + a_2 + a_3 = 2\pi$ although one of the three angles may still be zero. In either case, it holds that $a_1 + a_2 + a_3 \leq 2\pi$.

In the sequel, we first prove $a_1 = b_1$ and $a_3 = b_3$.

Note that $b_i \in (0, \pi)$ for each $i = 1, 2, 3$, and $a_i \in [0, 2\pi)$ for each $i = 1, 2, 3$.

Assume to the contrary that $b_1 \neq a_1$ and $b_3 \neq a_3$. Due to $\cos(b_1) = \cos(a_1)$, it must be that $a_1 = 2\pi - b_1$. Since $b_1 \in (0, \pi)$, we obtain $a_1 \in (\pi, 2\pi)$. Similarly, due to $\cos(b_3) = \cos(a_3)$, it must be that $a_3 = 2\pi - b_3$. Since $b_3 \in (0, \pi)$, we obtain $a_3 \in (\pi, 2\pi)$. This yields that $a_1 + a_3 > 2\pi$, contradicting with $a_1 + a_2 + a_3 \leq 2\pi$.

Therefore, at least one of $b_1 \neq a_1$ and $b_3 \neq a_3$ must be true. Without loss of generality, we assume $b_1 = a_1$.

Since $\frac{\sin(a_3)}{\sin(b_3)} = \frac{\sin(a_1)}{\sin(b_1)}$ and $b_1 = a_1$, it is obviously true that $\sin(a_3) = \sin(b_3)$. Recall that we already have $\cos(a_3) = \cos(b_3)$. Therefore, it must hold that $a_3 = b_3 \pmod{2\pi}$. Since $a_i \in [0, 2\pi)$ and $b_3 \in (0, \pi)$, we obtain that $a_3 = b_3$.

This proves that $a_1 = b_1$ and $a_3 = b_3$. Since neither b_1 nor b_3 is zero, we obtain $a_1 \neq 0$ and $a_3 \neq 0$, and thus $a_1 + a_2 + a_3 = 2\pi$. This implies that $a_2 = b_2$ since $b_1 + b_2 + b_3 = 2\pi$. The lemma thus follows. \square

5 Computing the Candidate Points

In this section, with the help of the observations in Sections 3 and 4, we compute a set S of candidate points such that all geodesic centers must be in S .

Let s be any geodesic center. Recall that $F(s)$ is the set of all farthest points of s . Depending on whether s is in \mathcal{V} , E , or \mathcal{I} , the size $|F(s)|$, whether some points of $F(s)$ are in \mathcal{V} , E , or \mathcal{I} , whether s has a degenerate farthest point, there are a significant (but still constant) number of cases. For each case, our algorithm uses an exhaustive-search approach to compute a set of candidate points such that s must be in the set. In particular, there are four cases, called *dominating cases*, for which the number of candidate points is $O(n^{11})$. But the total number of the candidate points for all other cases is only $O(n^{10})$. Therefore, the set S has a total of $O(n^{11})$ candidate points. We will show that S can be computed in $O(n^{11} \log n)$ time.

To find the geodesic centers in S , a straightforward algorithm works as follows. For each point $\hat{s} \in S$, we can compute $d_{\max}(\hat{s})$ in $O(n \log n)$ time by first computing the shortest path map $SPM(\hat{s})$ of \hat{s} in $O(n \log n)$ time [12] and then obtaining the maximum geodesic distance from \hat{s} to all vertices

of $SPM(\hat{s})$. Since all geodesic centers are in S , the points of S with the smallest $d_{\max}(\hat{s})$ are geodesic centers of \mathcal{P} .

Since $|S| = O(n^{11})$, the above algorithm runs in $O(n^{12} \log n)$ time. Let S_d denote the set of the candidate points for the four dominating cases. Clearly, the bottleneck is on finding the geodesic centers from S_d . To improve the algorithm, when we compute the candidate points of S_d , we will maintain the corresponding *path information*. By using these path information and based on new observations, we will present in Section 6 an $O(n^{11} \log n)$ time “pruning algorithm” that can eliminate most of the points from S_d such that none of the eliminated points is a geodesic center and the number of remaining points in S_d is only $O(n^{10})$. Consequently, we can use the above straightforward algorithm to find all geodesic centers in $O(n^{11} \log n)$ time.

In the rest of this section, we focus on computing the set S . Our algorithm for finding the geodesic centers from S (in particular, the pruning algorithm) will be given in Section 6.

In the following, we adopt the follow convention on notation: s represents a true geodesic center and \hat{s} represents a corresponding candidate point.

First of all, for the case $s \in \mathcal{V}$, we consider all polygon vertices of \mathcal{V} as candidate points. In the following, we only consider the case $s \in E$ or $s \in \mathcal{I}$. If s has a degenerate farthest point, we refer to it as the *degenerate case*; otherwise it is a *non-degenerate or general case*. We will compute candidate points for the general case and the degenerate case in Sections 5.2 and 5.3, respectively. The four dominating cases are all general cases. We begin with computing the candidate points for a special case in Section 5.1.

5.1 A Special Case

Consider a non-degenerate farthest point t of s such that t is in E or \mathcal{I} . If t is in E , then there are exactly two shortest s - t paths; we say that t is a *special* farthest point of s if the π -range $R_\pi(s, t)$ of s with respect to t and the two shortest paths is \emptyset (i.e., the special case in Lemma 8). Similarly, if t is in \mathcal{I} , then there are exactly three shortest s - t paths; we say that t is a *special* farthest point of s if the π -range $R_\pi(s, t)$ of s with respect to t and the three shortest paths is \emptyset (i.e., the special case in Lemma 11).

If s has a special farthest point, we refer to it as the *special case*. Let s be any geodesic center in the special case. In this subsection, we will compute a set S_1 of $O(n^5)$ candidate points in $O(n^5 \log n)$ time for the special case, such that s must be in S_1 .

Let t be a special farthest point of s . By the above definition, t is in E or \mathcal{I} . We discuss the two cases below.

The case $t \in E$. Let e be the polygon edge that contains t . Let $\pi_1(s, t)$ and $\pi_2(s, t)$ be the two shortest s - t paths. For each $i = 1, 2$, let u_i and v_i be the s -pivot and t -pivot of $\pi_i(s, t)$, respectively (i.e., $\pi_i(s, t) = \pi_{u_i, v_i}(s, t)$). Then, we have $|su_1| + d(u_1, v_1) + |v_1t| = |su_2| + d(u_2, v_2) + |v_2t|$. We define the angles $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the same way as those for Lemma 8 in Section 4.2. Since $R_\pi(s, t) = \emptyset$, by Lemma 8, $\beta_1 + \beta_2 = \pi$ and $\alpha_2 - \alpha_1 = \pm\pi$.

Note that $\alpha_2 - \alpha_1 = \pm\pi$ implies that s is on the line segment $\overline{u_1u_2}$. $\beta_1 + \beta_2 = \pi$ implies that l_t bisects the angle $\angle v_1tv_2$, where l_t is the line through t and perpendicular to e .

In summary, s and t satisfy the following “constraints”: (1) $t \in e$; (2) $|su_1| + d(u_1, v_1) + |v_1t| = |su_2| + d(u_2, v_2) + |v_2t|$; (3) $s \in \overline{u_1u_2}$; (4) l_t bisects the angle $\angle v_1tv_2$.

If we consider the coordinates of s and t as four variables, the above four (independent) constraints can determine s and t . An easy observation is that s must be in a bisector edge of two cells of $SPM(t)$ whose roots are u_1 and u_2 , respectively (in fact s is the intersection of the bisector edge and $\overline{u_1u_2}$). Correspondingly, we compute the candidate points in an exhaustive way as follows.

We enumerate all possible combinations of a polygon edge as e and two polygon vertices as v_1 and v_2 . For each combination, we find a point t on e such that the above condition (4) can be satisfied.

Next we compute the shortest path map $SPM(t)$ of t . For each bisector edge of $SPM(t)$, we consider it as e and let u_1 and u_2 be the roots of the two cells of $SPM(t)$ incident to e ; we report the intersection $e \cap \overline{u_1 u_2}$ (if any) as a candidate point \hat{s} . In this way, we can compute at most $O(n)$ candidate points on $SPM(t)$ since the combinatorial complexity of $SPM(t)$ is $O(n)$. Thus, for each combination of e , v_1 , and v_2 , we can compute $O(n)$ candidate points. Since there are $O(n^3)$ combinations, we can compute $O(n^4)$ candidate points and we add these points to S_1 . By our above discussions, the geodesic center s must be in S_1 .

Since computing a shortest path map takes $O(n \log n)$ time [12], the running time of the above algorithm is bounded by $O(n^4 \log n)$.

The case $t \in \mathcal{I}$. In this case, there are exactly three shortest s - t paths: $\pi_i(s, t) = \pi_{u_i, v_i}(s, t)$ with $i = 1, 2, 3$. Then we have $|su_1| + d(u_1, v_1) + |v_1 t| = |su_2| + d(u_2, v_2) + |v_2 t| = |su_3| + d(u_3, v_3) + |v_3 t|$. We define the angles a_i, b_i , for $i = 1, 2, 3$, in the same way as those for Lemma 11 in Section 4.4. By Lemma 11, $a_i = b_i$ for $i = 1, 2, 3$.

If we consider the coordinates of s and t as four variables, the above equation on the lengths of the three shortest paths provide two constraints and the identities of the three pairs of angles provide other two (independent) constraints, and thus the total four (independent) constraints can determine s and t . Correspondingly, we compute the candidate points for s as follows.

We enumerate all possible combinations of three polygon vertices as v_1, v_2, v_3 . We compute the shortest path maps of v_1, v_2 , and v_3 in $O(n \log n)$ time. Next we compute the overlay of the three shortest path maps. The overlay is of size $O(n^2)$ and can compute in $O(n^2 \log n)$ time [3, 7]. Then, for each cell of the overlay, we obtain the three roots of the cell in the three shortest path maps and consider them as u_1, u_2, u_3 . Finally, we use the above four constraints to determine a constant number of pairs (\hat{s}, t) (we assume this can be done in constant time since the angles a_i and b_i can be parameterized by the coordinates of \hat{s} and t), and we add each \hat{s} to S_1 as a candidate point. In this way, for each combination of v_1, v_2, v_3 , we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^3)$ combinations, we can compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

5.2 The General Case

Recall that in the general case s does not have a degenerate farthest point. Depending on whether s is in \mathcal{I} or in E , there are two main cases. We first consider the case $s \in \mathcal{I}$.

The case $s \in \mathcal{I}$. Depending on whether $|F(s)|$ is 1, 2, or at least 3, there are further three cases.

The case $|F(s)| = 1$. If $|F(s)| = 1$, let t be the only farthest point of s . Thus, $R(s) = R(s, t)$. By Lemma 1, $R(s) = \emptyset$. Hence, $R(s, t) = \emptyset$. Since $s \in \mathcal{I}$, by our observations in Section 4 (in particular, Lemmas 6, 7, 8, 9, 11, 12), t is either in E or \mathcal{I} , and in either case $R_\pi(s, t) = \emptyset$ since $s \in \mathcal{I}$ and $R_f(s)$ consists of all directions. Hence, t must be a special farthest point of s , which implies that s has been computed in the candidate point set S_1 .

The case $|F(s)| = 2$. If $|F(s)| = 2$, we assume that $F(s)$ does not have any special farthest points since otherwise s would have already been computed in S_1 . Let $F(s) = \{t_1, t_2\}$. Depending on whether each of t_1 and t_2 is in \mathcal{V} , E , or \mathcal{I} , there are several cases. In the following, we use (x, y, z) to refer to the case where x, y, z points of $F(s)$ are in \mathcal{I}, E , and \mathcal{V} , respectively, with $x + y + z = 2$. For example, $(1, 1, 0)$ refers to the case where one point of $F(s)$ is in \mathcal{I} and the other is in E .

1. *Case $(2, 0, 0)$.* We first consider the most general case where $t_i \in \mathcal{I}$ for $i = 1, 2$. Other cases are very similar. For each $i = 1, 2$, since $t_i \in \mathcal{I}$, t_i has three shortest paths to s : $\pi_{u_{ij}, v_{ij}}(s, t_i)$ with

$j = 1, 2, 3$. Hence, we have the following

$$\begin{aligned} |t_1 v_{11}| + d(v_{11}, u_{11}) + |u_{11} s| &= |t_1 v_{12}| + d(v_{12}, u_{12}) + |u_{12} s| = |t_1 v_{13}| + d(v_{13}, u_{13}) + |u_{13} s| \\ &= |t_2 v_{21}| + d(v_{21}, u_{21}) + |u_{21} s| = |t_2 v_{22}| + d(v_{22}, u_{22}) + |u_{22} s| = |t_2 v_{23}| + d(v_{23}, u_{23}) + |u_{23} s|. \end{aligned}$$

Further, since s has only two farthest points and $s \in \mathcal{I}$, it must hold that $R(s) = R_\pi(s, t_1) \cap R_\pi(s, t_2) = \emptyset$ by Lemma 1. Recall that we have assumed that $F(s)$ does not have any special farthest points. Therefore, neither $R_\pi(s, t_1)$ nor $R_\pi(s, t_2)$ is \emptyset . By Lemma 11, each of $R_\pi(s, t_1)$ and $R_\pi(s, t_2)$ is an open range of size π . For each $i = 1, 2$, the directions of $R_\pi(s, t_i)$ are delimited by an open half-plane whose bounding line contains s , and we refer to the bounding line as the *bounding line* of the range $R_\pi(s, t_i)$. Since $R_\pi(s, t_1) \cap R_\pi(s, t_2) = \emptyset$, we can obtain that the bounding line of $R_\pi(s, t_1)$ is the same as that of $R_\pi(s, t_2)$.

If we consider the coordinates of s , t_1 , and t_2 as six variables, the above system of equations on the distances of the six shortest paths provide five constraints and that the bounding lines of the two π -ranges overlap provides another constraint (note that the bounding lines of the two π -range can be parameterized by the coordinates of s , t_1 , and t_2). Hence, the six (independent) constraints can determine the triple (s, t_1, t_2) . Correspondingly, our algorithm for computing the candidate points works as follows.

We enumerate all possible combinations of six polygon vertices as v_{ij} for $i = 1, 2$ and $j = 1, 2, 3$. We compute overlay of the shortest path maps of the six vertices. For each cell of the overlay, we obtain the six roots in the six shortest path maps as u_{ij} for $i = 1, 2$ and $j = 1, 2, 3$. Using the six constraints as discussed above, we can obtain a constant number of triples (\hat{s}, t_1, t_2) , and each such \hat{s} is a candidate point.

In this way, for each combination, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^6)$ combinations, we can compute $O(n^8)$ candidate points in overall $O(n^8 \log n)$ time. As analyzed above, s must be one of these candidate points.

2. *Case (1, 1, 0).* In this case, one of t_1 and t_2 is in \mathcal{I} and the other is in E . Without loss of generality, we assume $t_1 \in \mathcal{I}$ and $t_2 \in E$. Hence, t_1 has three shortest paths to s : $\pi_{u_{1j}, v_{1j}}(s, t_1)$ for $j = 1, 2, 3$, and t_2 has two shortest paths to s : $\pi_{u_{2j}, v_{2j}}(s, t_2)$ for $j = 1, 2$. Hence, we have the following

$$\begin{aligned} |t_1 v_{11}| + d(v_{11}, u_{11}) + |u_{11} s| &= |t_1 v_{12}| + d(v_{12}, u_{12}) + |u_{12} s| = |t_1 v_{13}| + d(v_{13}, u_{13}) + |u_{13} s| \\ &= |t_2 v_{21}| + d(v_{21}, u_{21}) + |u_{21} s| = |t_2 v_{22}| + d(v_{22}, u_{22}) + |u_{22} s|. \end{aligned}$$

As in the previous case, $R_\pi(s, t_1) \cap R_\pi(s, t_2) = \emptyset$ and each of the two π -ranges is nonempty; further, and the two bounding lines of these two π -ranges must overlap.

Let e be the polygon edge that contains t .

The above equations provide four constraints and the overlap of the two bounding lines of the two π -ranges provides another constraint. In addition, $t_2 \in e$ gives the sixth constraint. As in the previous case, the six constraints can determine s , t_1 , and t_2 if we consider their coordinates as six variables. Correspondingly, we compute candidate points as follows.

We enumerate all possible combinations of a polygon edge as e and five polygon vertices as $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}$. We compute the overlay of the shortest path maps of the five vertices. For each cell of the overlay, we obtain the five roots in the five shortest path maps as $u_{11}, u_{12}, u_{13}, u_{21}, u_{22}$. Then we obtain the equations as above. Along with the constraint that $t_2 \in e$ and the constraint that the two bounding lines of the π -ranges $R_\pi(\hat{s}, t_1)$ and $R_\pi(\hat{s}, t_2)$ overlap, we can obtain a constant number of triples (\hat{s}, t_1, t_2) , and each such \hat{s} is a candidate point. In this way, for each combination, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^8)$ combinations, we can compute $O(n^8)$ candidate points for s in $O(n^8 \log n)$ time.

3. *Other Cases.* The other cases are all very similar (i.e., $(1, 0, 1)$, $(0, 2, 0)$, $(0, 1, 1)$, $(0, 0, 2)$), and we can compute a total of $O(n^8)$ candidate points in $O(n^8 \log n)$ time. We omit the details.

As a summary, for the case $|F(s)| = 2$, we can compute at most $O(n^8)$ candidate centers in $O(n^8 \log n)$ time.

The case $|F(s)| \geq 3$. In this case, the geodesic center s has at least three farthest points. We assume s does not have any special farthest points since otherwise s would have already been computed in S_1 . Hence, for each $t \in F(s)$, the π -range $R_\pi(s, t)$ is not empty and thus is an open range of size π . Together with Lemma 1, this further implies that s must have three farthest points t_1, t_2, t_3 such that $R_\pi(s, t_1) \cap R_\pi(s, t_2) \cap R_\pi(s, t_3) = \emptyset$. Depending on whether each of t_i for $i = 1, 2, 3$ is in $\mathcal{I}, E, \mathcal{V}$, there are several cases. Similarly, we use (x, y, z) to refer to the case where x, y , and z points of t_1, t_2, t_3 are in \mathcal{I}, E , and \mathcal{V} , respectively, with $x + y + z = 3$.

We begin our discussion with the most general case $(3, 0, 0)$, where t_1, t_2, t_3 are all in \mathcal{I} . This is one of the four dominating cases and we will need to compute $O(n^{11})$ candidate points. Further, our algorithm is slightly different from the previous cases in the following sense: First, in addition to the candidate points, we will also maintain the corresponding *path information*; second, when computing the candidate points, we will have a “validation procedure”. Recall that S_d is the set of candidate points for all four dominating cases. The path information will be used in the next section to quickly prune most of the points in S_d and the validation procedure can be considered as a “preliminary pruning step”. The details are given below.

For each $i = 1, 2, 3$, since t_i is in \mathcal{I} , there are exactly three shortest paths from s to t_i : $\pi_{u_{ij}, v_{ij}}(s, t)$ with $j = 1, 2, 3$. Hence, we have the following

$$\begin{aligned} |t_1 v_{11}| + d(v_{11}, u_{11}) + |u_{11} s| &= |t_1 v_{12}| + d(v_{12}, u_{12}) + |u_{12} s| = |t_1 v_{13}| + d(v_{13}, u_{13}) + |u_{13} s| \\ &= |t_2 v_{21}| + d(v_{21}, u_{21}) + |u_{21} s| = |t_2 v_{22}| + d(v_{22}, u_{22}) + |u_{22} s| = |t_2 v_{23}| + d(v_{23}, u_{23}) + |u_{23} s| \\ &= |t_3 v_{31}| + d(v_{31}, u_{31}) + |u_{31} s| = |t_3 v_{32}| + d(v_{32}, u_{32}) + |u_{32} s| = |t_3 v_{33}| + d(v_{33}, u_{33}) + |u_{33} s|. \end{aligned}$$

If we consider the coordinates of s, t_1, t_2 , and t_3 as eight variables, the above equations on the lengths of nine paths provide eight constraints, which are sufficient to determine all four points. Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of nine polygon vertices as v_{i1}, v_{i2}, v_{i3} , with $i = 1, 2, 3$. For each combination, we compute the overlay of the shortest path maps of these nine vertices. The size of the overlay is $O(n^2)$. For each cell C of the overlay, we obtain nine roots of the shortest path maps and consider them as u_{i1}, u_{i2}, u_{i3} for $i = 1, 2, 3$. We form the above system of eight equations and solve it to obtain a constant number of quadruples of points $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$. We also obtain a path length, denoted by $d(\hat{s})$, which is equal to the value in the above equations, e.g., $d(\hat{s}) = |\hat{s} u_{11}| + d(u_{11}, v_{11}) + |v_{11} \hat{t}_1|$. In addition, we perform a validation procedure on each such quadruple $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ as follows.

First, we check whether \hat{s} is in C , which can be done in $O(\log n)$ time by using a point location data structure [9,14] with $O(n^2)$ time and space preprocessing on the overlay. If yes, for each t_i with $i = 1, 2, 3$, we check whether $d(\hat{s})$ is equal to $d(\hat{s}, \hat{t}_i)$, which can be computed in $O(\log n)$ time by using the two-point shortest path query data structure given by Chiang and Mitchell [7] with $O(n^{11})$ time and space preprocessing on \mathcal{P} . If yes, we check whether v_{i1}, v_{i2}, v_{i3} satisfy the condition in Observation 1(1), i.e., whether \hat{t}_i is in the interior of the triangle $\triangle v_{i1} v_{i2} v_{i3}$ for each $i = 1, 2, 3$. If yes, for each \hat{t}_i with $i = 1, 2, 3$, we check whether the order of the vertices of v_{i1}, v_{i2}, v_{i3} around \hat{t}_i are consistent with the order of the vertices of u_{i1}, u_{i2}, u_{i3} (we say that the two orders are *consistent* if after reordering the indices, v_{i1}, v_{i2}, v_{i3} are clockwise around \hat{t}_i while u_{i1}, u_{i2}, u_{i3} are counterclockwise around s ; note that this consistency is needed for determining the π -range in Lemma 11). If yes, for each \hat{t}_i with $i = 1, 2, 3$, we compute the π -range $R_\pi(s, \hat{t}_i)$ determined by Lemma 11, and then check whether $R_\pi(\hat{s}, \hat{t}_1) \cap R_\pi(\hat{s}, \hat{t}_2) \cap R_\pi(\hat{s}, \hat{t}_3)$ is empty. If yes, we say that the quadruple $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ *passes* the validation procedure and we call \hat{s} a *valid* candidate point and add \hat{s} to the set S_d . In addition, we maintain the following *path information*: $d(\hat{s})$, \hat{t}_i , v_{ij} , and u_{ij} , with $1 \leq i \leq 3$ and $1 \leq j \leq 3$. In fact, only $d(\hat{s})$ will be used later in the algorithm and all other information are only for the reference purpose in the analysis.

In this way, for each combination of nine polygon vertices, we can compute $O(n^2)$ valid candidate for S_d in $O(n^2 \log n)$ time (not including the preprocessing time). Since there are $O(n^9)$ combinations, we can compute a total of $O(n^{11})$ valid candidate points for S_d in $O(n^{11} \log n)$ time.

Note that the geodesic center s and the quadruple (s, t_1, t_2, t_3) discussed above must pass the validation procedure, and thus the quadruple (s, t_1, t_2, t_3) will be computed by our exhaustive-search algorithm and s will be computed as a valid candidate point in S_d .

Based on our validation procedure, the following observation summarizes the properties of the valid candidate points. These properties will be used to prove the correctness of our pruning algorithm in Section 6.

Observation 6 *Suppose $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ is a quadruple that passes the validation procedure, with u_{ij} and v_{ij} , $i = 1, 2, 3$ and $j = 1, 2, 3$ defined as above. Then the following hold.*

1. For each $i = 1, 2, 3$, $\overline{\hat{s}u_{ij}} \cup \pi(u_{ij}, v_{ij}) \cup \overline{v_{ij}\hat{t}_i}$ is a shortest path from \hat{s} to \hat{t}_i for each $j = 1, 2, 3$.
2. For each $i = 1, 2, 3$, v_{i1}, v_{i2}, v_{i3} satisfy the condition of Observation 1(1), i.e., \hat{t}_i is in the interior of the triangle $\triangle v_{i1}v_{i2}v_{i3}$.
3. $d(\hat{s}) = d(\hat{s}, \hat{t}_i)$ for each $i = 1, 2, 3$.
4. $R_\pi(\hat{s}, \hat{t}_1) \cap R_\pi(\hat{s}, \hat{t}_2) \cap R_\pi(\hat{s}, \hat{t}_3) = \emptyset$.

The above computes the candidate points for the case $(3, 0, 0)$. In the following, we compute candidate points for other cases. The algorithms are similar.

1. *Case $(2, 1, 0)$.* In this case, two of t_1, t_2, t_3 are in \mathcal{I} and the third one is in E . Without loss of generality, we assume t_1 and t_2 are in \mathcal{I} and t_3 is in E . Let e be the polygon edge containing t_3 . This is the second dominating case. We will also perform a validation procedure and maintain the corresponding path information.

For each $i = 1, 2$, since t_i is in \mathcal{I} , there are three shortest paths from s to t_i : $\pi_{u_{ij}, v_{ij}}(s, t)$ for $j = 1, 2, 3$. Since $t_3 \in E$, there are two shortest paths from s to t_3 : $\pi_{u_{3j}, v_{3j}}(s, t)$ for $j = 1, 2$. Hence, we have the following

$$\begin{aligned} & |t_1v_{11}| + d(v_{11}, u_{11}) + |u_{11}s| = |t_1v_{12}| + d(v_{12}, u_{12}) + |u_{12}s| = |t_1v_{13}| + d(v_{13}, u_{13}) + |u_{13}s| \\ & = |t_2v_{21}| + d(v_{21}, u_{21}) + |u_{21}s| = |t_2v_{22}| + d(v_{22}, u_{22}) + |u_{22}s| = |t_2v_{23}| + d(v_{23}, u_{23}) + |u_{23}s| \\ & = |t_3v_{31}| + d(v_{31}, u_{31}) + |u_{31}s| = |t_3v_{32}| + d(v_{32}, u_{32}) + |u_{32}s|. \end{aligned}$$

If we consider the coordinates of s, t_1, t_2, t_3 as eight variables, the above equations give seven constraints. With the additional constraint that t_3 is on e , we can determine (s, t_1, t_2, t_3) . Correspondingly, we compute the candidate points as follows.

We enumerate all combinations of eight polygon vertices (as $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, v_{31}, v_{32}$) and one polygon edge (as e). For each combination, we can compute $O(n^2)$ quadruples $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$. For each such quadruple, we also perform the validation procedure similarly as before with the following differences. For \hat{t}_3 , since there is no vertex v_{33} , we ignore all operations that involve v_{33} . Further, we check whether $\{v_{11}, v_{12}\}$ satisfy the condition in Observation 1(2) (instead of Observation 1(1)). We determine the range $R_\pi(s, t_3)$ by using Lemma 8 (instead of Lemma 11).

In this way, we can compute at most $O(n^{11})$ valid candidate points in $O(n^{11} \log n)$ time and add them to S_d , and also, we maintain the corresponding path information similarly as before. Observation 6 still holds correspondingly with the following difference: in (1) there are only two shortest paths for \hat{t}_3 ; in (2) for \hat{t}_3 , $\{v_{11}, v_{12}\}$ satisfy the condition of Observation 1(2).

2. *Cases $(1, 2, 0)$ and $(0, 3, 0)$.* These are the other two dominating cases. We can compute $O(n^{11})$ valid candidate points in $O(n^{11} \log n)$ time. The algorithms are very similar to the previous two cases. For each case, we also need a validation procedure and keep the path information. We omit the details.

We have discussed all four dominating cases. Note that the candidate points for all four dominating cases are in \mathcal{I} . The rest of the cases are not dominating cases. For each of the remaining cases, the validation procedure and the path information will not be needed any more.

3. *Other cases.* Our algorithms for all other cases (e.g. $(2, 0, 1)$, $(1, 1, 1)$, $(1, 0, 2)$, $(0, 2, 1)$, $(0, 1, 2)$, $(0, 0, 3)$) are very similar as before (except that the validation procedure and the path information are not needed). We can compute a total of $O(n^{10})$ candidate points in $O(n^{10} \log n)$ time. We omit the details.

As a summary for the case $s \in \mathcal{I}$, for the four dominating cases, we have computed $O(n^{11})$ valid candidate points in $O(n^{11} \log n)$ time with the corresponding path information. For all other cases, we have computed $O(n^{10})$ candidate points.

The case $s \in E$. Our algorithm for computing candidate points for this case is slightly different. Consider a geodesic center s in E . Let e_s denote the polygon edge that contains s . Depending on whether $|F(s)|$ is 1 or not, there are two cases.

The case $|F(s)| = 1$. If $|F(s)| = 1$, let t be the only farthest point of s . We assume that t is not a special farthest point of s since otherwise s would have already been computed in S_1 in Section 5.1. Thus, by Lemma 8, the π -range $R_\pi(s, t)$ is an open range of size π . Since t is the only farthest point of s , by Lemma 1, $R(s) = R(s, t) = \emptyset$. By Lemma 9, $R(s, t) = \emptyset$ implies that the bounding line of the range $R_\pi(s, t)$ contains e_s , and this provides a constraint for determining s and t . Depending on whether t is in \mathcal{I} , E , or \mathcal{V} , there are three cases.

1. *The case $t \in \mathcal{I}$.* We first consider the most general case in which $t \in \mathcal{I}$. There are three shortest s - t paths: $\pi_{u_i v_i}(s, t)$ for $i = 1, 2, 3$. Thus, we have the following

$$|tv_1| + d(v_1, u_1) + |u_1 s| = |tv_2| + d(v_2, u_2) + |u_2 s| = |tv_3| + d(v_3, u_3) + |u_3 s|.$$

If we consider the coordinates of s and t as four variables, then the above equations provide two constraints and $s \in e_s$ gives another constraint. In addition, that the bounding line of the range $R_\pi(s, t)$ contains e_s provides the fourth constraint. Hence, the four constraints can determine s and t . Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of three polygon vertices as v_1, v_2, v_3 . For each combination, we compute the overlay of the shortest path maps of v_1, v_2 , and v_3 . For each cell C of the overlay, if it contains a portion of a polygon edge, we consider the edge as e_s and obtain three roots of the shortest path maps as u_1, u_2 , and u_3 . Using the four constraints discussed above, we can obtain a constant number of tuples (\hat{s}, t) and each such \hat{s} is a candidate point. In this way, we can compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

2. *The case $t \in E$.* Let e_t be the polygon edge that contains t . There are two shortest s - t paths, which gives one constraint for determining s and t . $s \in e_s$ and $t \in e_t$ provide two constraints. Finally, that the bounding line of the range $R_\pi(s, t)$ contains e_s gives the fourth constraint. Hence, we still have four constraints to determine s and t .

The algorithm is similar to the previous case. We can compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time. We omit the details.

3. *The case $t \in \mathcal{V}$.* In this case, t is a polygon vertex, which provides two constraints (or t is fixed already). $s \in e_s$ is another constraint. As before, that the bounding line of the range $R_\pi(s, t)$ contains e_s provides the fourth constraint. We compute the candidate points as follows.

We enumerate all polygon vertices. For each vertex, we consider it as t and compute the shortest path map $SPM(t)$. For each cell C of $SPM(t)$, if it contains a portion of polygon edge on its boundary, then we consider that edge as e_s . Next, we find a candidate point \hat{s} on that portion of e_s in C such that the π -range $R_\pi(\hat{s}, t)$ (determined by Lemma 8) has its bounding line contain e_s . In this way, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time.

In summary, for the case $|F(s)| = 1$, we compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

The case $|F(s)| \geq 2$. Let t_1 and t_2 be the two farthest points of s . We assume s does not have any special farthest points since otherwise s would have already been computed in S_1 . Then, neither $R_\pi(s, t_1)$ nor $R_\pi(s, t_2)$ is empty. Depending on whether each of t_1 and t_2 is in \mathcal{I} , E , or \mathcal{V} , there are several cases.

We first consider the most general case where both t_1 and t_2 are in \mathcal{I} . Other cases are very similar. For each $i = 1, 2$, since $t_i \in \mathcal{I}$, there are three shortest paths from s to t_i : $\pi_{u_{ij}, v_{ij}}(s, t)$ with $j = 1, 2, 3$. Hence, we have the following

$$\begin{aligned} |t_1 v_{11}| + d(v_{11}, u_{11}) + |u_{11} s| &= |t_1 v_{12}| + d(v_{12}, u_{12}) + |u_{12} s| = |t_1 v_{13}| + d(v_{13}, u_{13}) + |u_{13} s| \\ &= |t_2 v_{21}| + d(v_{21}, u_{21}) + |u_{21} s| = |t_2 v_{22}| + d(v_{22}, u_{22}) + |u_{22} s| = |t_2 v_{23}| + d(v_{23}, u_{23}) + |u_{23} s|. \end{aligned}$$

The above equations provide five constraints for s and t . With the constraint that $s \in e_s$, we can determine s , t_1 , and t_2 , if we consider their coordinates as six variables. Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of six polygon vertices as $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}$. For each combination, we compute the overlay of the shortest path maps of the six vertices. For each cell C of the overlay, if C contains a portion of a polygon edge, then we consider the edge as e_s and using the six constraints mentioned above to compute at most a constant number of triples (\hat{s}, t_1, t_2) . Hence, for each combination, we can compute at most $O(n^2)$ candidate points in $O(n^2 \log n)$ time. In this way, we can compute $O(n^8)$ candidate points in $O(n^8 \log n)$ time.

Candidate points for other cases can be computed similarly. We can compute $O(n^8)$ candidate points in $O(n^8 \log n)$ time. We omit the details.

As a summary for the case $s \in E$, we can compute a total of $O(n^8)$ candidate points in $O(n^8 \log n)$ time.

5.3 The Degenerate Case

In the degenerate case, the geodesic center s has at least one degenerate farthest point. Depending on whether s is in \mathcal{I} or E , there are two main cases.

The Case $s \in \mathcal{I}$. Depending on whether $|F(s)|$ is 1 or not, there are further two cases.

The case $|F(s)| = 1$. Let t be the only point of $F(s)$. Then t is a degenerate farthest point of s . Depending on whether t is in \mathcal{V} , E , or \mathcal{I} , there are three cases.

1. If $t \in \mathcal{V}$, then since t is degenerate, there are at least two shortest s - t paths. Further, since t is polygon vertex and due to our general position assumption that no two polygon vertices have more than one shortest path, it holds that $|U_s(t)| \geq 2$.

If $|U_s(t)| \geq 3$, then s is a vertex of $SPM(t)$. Correspondingly, we can compute candidate points as follows. For each polygon vertex, we consider it as t and compute its shortest path map $SPM(t)$. Then, we report each vertex of $SPM(t)$ as a candidate points. In this way, we can compute $O(n^2)$ candidate points for s in $O(n^2 \log n)$ time.

If $|U_s(t)| = 2$, let $U_s(t) = \{u_1, u_2\}$. We first prove a claim: $s \in \overline{u_1 u_2}$.

Indeed, suppose to the contrary that the claim is not true. Then, $\overline{s u_1}$ and $\overline{s u_2}$ form an angle $\angle u_1 s u_2 \in (0, \pi)$. Consider the direction r_s for moving s along the bisector of $\angle v_1 s v_2$ and towards the interior of $\angle v_1 s v_2$. If we move s along r_s with unit speed, since $\angle u_1 s u_2$ is strictly less than π and $U_s(t) = \{u_1, u_2\}$, each vertex $v \in U_t(s)$ has a coupled s -pivot $u \in U_s(t)$ with $d'_{u,v}(s, t) < 0$. Since $s \in \mathcal{I}$, r_s is a free direction. By Lemma 4, r_s is an admissible direction of s with respect

to t , i.e., $r_s \in R(s, t)$. Since $R(s) = R(s, t)$, we obtain that $R(s)$ is not empty, contradicting with Lemma 1. The claim thus follows.

$s \in \overline{u_1 u_2}$ provides a constraint for determining s (if we consider $t \in \mathcal{V}$ as fixed at a polygon vertex). Further, there are two shortest s - t paths $\pi_{u_i v_i}(s, t)$ with $i = 1, 2$. Hence, s must be in the bisector edge that is incident to the two cells whose roots are u_1 and u_2 , respectively, in the shortest path map $SPM(t)$. Correspondingly, we compute the candidate points as follows.

We enumerate all polygon vertices. For each polygon vertex, we consider it as t and compute $SPM(t)$. For each bisector edge of $SPM(t)$, we obtain the roots of the two cells incident to the bisector edge as u_1 and u_2 . Then, we compute a candidate point \hat{s} on the bisector edge such that $\hat{s} \in \overline{u_1 u_2}$ (i.e., \hat{s} is the intersection of $\overline{u_1 u_2}$ and the bisector edge). In this way, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time.

2. If $t \in E$, then since t is degenerate, there are at least three shortest s - t paths. Let e_t be the polygon edge that contains t . Since t is a farthest point of s , t must have two vertices v_1 and v_2 that satisfy the condition in Observation 1. Hence, there are two shortest s - t paths $\pi_{u_i, v_i}(s, t)$ for $i = 1, 2$.

Since $s \in \mathcal{I}$, we claim that the π -range $R_\pi(s, t)$ with respect to the two shortest paths $\pi_{u_i, v_i}(s, t)$ for $i = 1, 2$ must be empty. Assume to the contrary that this is not true. Then, by Lemma 5(2), $R(s, t)$ is not empty (in fact, $R_\pi(s, t) \subseteq R(s, t)$). Since t is the only farthest point of s , we obtain that $R(s) = R(s, t) \neq \emptyset$, which contradicts with Lemma 1.

Since $R_\pi(s, t) = \emptyset$, one can check that s has already been computed in S_1 in Section 5.1 (for the case $t \in E$).

3. If $t \in \mathcal{I}$, since t is degenerate, there are at most four shortest s - t paths. By Observation 1, $|U_t(s)| \geq 3$. Depending on whether $|U_t(s)| = 3$, there are two cases.

We first discuss the case where $|U_t(s)| = 3$. Let v_1, v_2, v_3 be the three vertices in $U_t(s)$. By Observation 1, t is in the interior of the triangle $\triangle v_1 v_2 v_3$. Then, there must exist three shortest s - t paths $\pi_{u_i, v_i}(s, t)$ such that no two paths cross each other, and this implies that the three paths are canonical. Let $R_\pi(s, t)$ denote the π -range of s with respect to t and the above three shortest paths.

If $R_\pi(s, t) = \emptyset$, then one can check that s has already been computed in S_1 in Section 5.1. Otherwise, as the above second case, since $s \in \mathcal{I}$ and $F(s) = \{t\}$, by Lemma 5(1) any direction in $R_\pi(s, t)$ is in $R(s, t)$, implying that $R(s) = R(s, t) \neq \emptyset$, which contradicts with Lemma 1.

Next we discuss the case where $|U_t(s)| > 3$.

By Observation 1, t is in the interior of the convex hull of all vertices of $U_t(s)$. If there exist three vertices v_1, v_2, v_3 of $U_t(s)$ such that t is in the interior of $\triangle v_1 v_2 v_3$, then we can still use the same approach as the above case for $|U_t(s)| = 3$. Otherwise, there must exist four vertices $v_1, v_2, v_3, v_4 \in U_t(s)$ such that t is the intersection of the two line segments $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$.

On the other hand, since $s \in \mathcal{I}$ and t is the only farthest point of s , we claim that $|U_s(t)| \geq 2$. Indeed, suppose to the contrary that $|U_s(t)| = 1$. Then, if we move s towards the only vertex u of $U_s(t)$ with unit speed, then for each vertex $v \in U_t(s)$, $d'_{u,v}(s, t) < 0$ holds. Hence, by Lemma 2(1), the above direction for s is in $R(s, t)$, implying that $R(s) = R(s, t) \neq \emptyset$, which contradicts with Lemma 1. Therefore, the claim follows.

Depending on whether $|U_s(t)| \geq 3$, there are two cases.

- (a) If $|U_s(t)| \geq 3$, then s is a vertex of the shortest path map $SPM(t)$. Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of four polygon vertices as v_i for $1 \leq i \leq 4$. For each such combination, we compute the intersection of $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$ and consider the intersection as t . Then we compute $SPM(t)$ and return all vertices of $SPM(t)$ as candidate points. In this way, for each combination, we can compute at most $O(n)$ candidate points in $O(n \log n)$ time. Since there are $O(n^4)$ combinations, we can compute a total of $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

- (b) If $|U_s(t)| = 2$, let $U_s(t) = \{u_1, u_2\}$. An easy observation is that s must be on the bisector edge of u_1 and u_2 in $SPM(t)$. Further, by the same analysis as before, we can show that s must be on $\overline{u_1 u_2}$. Correspondingly, we can compute the candidate points as follows.

We enumerate all possible combinations of four polygon vertices as v_i for $1 \leq i \leq 4$. For each such combination, we compute the intersection of $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$ and consider the intersection as t . Then, we compute $SPM(t)$. For each bisector edge of $SPM(t)$, we obtain the roots of the two cells incident to the bisector edge as u_1 and u_2 , respectively. Then, we find the intersection of $\overline{u_1 u_2}$ and the bisector edge as a candidate point. Since there are $O(n)$ bisector edges, we can find $O(n)$ candidate points. In this way, we can compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

This finishes the case for $|F(s)| = 1$.

The case $|F(s)| \geq 2$. Recall that s has at least one degenerate farthest point. Let t_1 and t_2 be any two points of $F(s)$ such that t_1 is degenerate.

We first consider the most general case where both t_1 and t_2 are in \mathcal{I} . For each $i = 1, 2$, t_i has at least three shortest paths from s and $|U_{t_i}(s)| \geq 3$. Since t_1 is degenerate, t_1 has at least four shortest paths from s . Depending on whether $|U_{t_1}(s)| = 3$, there are two cases.

If $|U_{t_1}(s)| > 3$, let v_{1j} for $1 \leq j \leq 4$ be any four vertices of $U_{t_1}(s)$. Hence, there are four shortest paths from s to t_1 : $\pi_{u_{1j}, v_{1j}}(s, t_1)$ with $1 \leq j \leq 4$. For t_2 , let v_{2j} for $1 \leq j \leq 3$ be any three vertices of $U_{t_2}(s)$, and there are three shortest paths from s to t_2 : $\pi_{u_{2j}, v_{2j}}(s, t_2)$ with $1 \leq j \leq 3$. We can form a system of equations with the lengths of the above seven shortest paths, which give six constraints to determine (s, t_1, t_2) . Correspondingly, we can compute $O(n^9)$ candidate points in $O(n^9 \log n)$ time. The algorithm is similar to the previous algorithms and we omit the details.

If $|U_{t_1}(s)| = 3$, let v_{1j} for $1 \leq j \leq 3$ be the three vertices of $U_{t_1}(s)$. Since there are at least four shortest paths from s to t_1 and due to our general position assumption that no two polygon vertices have more than one shortest path, there exists a vertex, say v_{11} , in $U_{t_1}(s)$ such that there are at least two shortest paths from s to v_{11} . Hence, s must be on a bisector edge of the shortest path map $SPM(v_{11})$.

Let $\pi_{u_{1j}, v_{1j}}(s, t_1)$ for $j = 1, 2, 3$ be three shortest paths from s to t_1 . Let $\pi_{u_{2j}, v_{2j}}(s, t_2)$ for $j = 1, 2, 3$ be three shortest paths from s to t_2 . We can form a system of equations with the lengths of the six paths. These equations can give five constraints, and along with that s is on a bisector edge of $SPM(t_1)$, we can determine (s, t_1, t_2) (equivalently, one may also consider that the seven shortest paths give six constraints). Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of six polygon vertices as v_{ij} for $i = 1, 2$ and $j = 1, 2, 3$. For each combination, we compute the overlay of the shortest path maps of these vertices. For each cell of the overlay, if its boundary has a portion of a bisector edge of $SPM(v_{11})$, then based on the above system of equations on the lengths of the six paths, we determine a constant number of triples (\hat{s}, t_1, t_2) and each such \hat{s} is a candidate point. Since the size of the overlay is $O(n^2)$, we can determine $O(n^2)$ candidate points for each combination. Since there are $O(n^6)$ combinations, we can compute $O(n^8)$ candidate points in $O(n^8 \log n)$ time.

The above computes $O(n^9)$ candidate points for the case where both t_1 and t_2 are in \mathcal{I} .

Other cases are very similar. For example, consider the case where $t_1 \in \mathcal{I}$ and t_2 is on a polygonal edge $e \in E$. Comparing with the previous case, since t_2 is in E , there will be one less constraint on the equations of shortest path lengths, but $t_2 \in e$ gives one more constraint. All other cases are similar. We can compute at most $O(n^9)$ candidate points in $O(n^9 \log n)$ for all other cases. The details are omitted.

This finishes the case for $|F(s)| \geq 2$ and thus the case for $s \in \mathcal{I}$.

The Case $s \in E$. Let e_s be the polygon edge that contains s . Let t be a degenerate farthest point of s . Depending on whether t is in \mathcal{V} , E , or \mathcal{I} , there are three cases.

1. If $t \in \mathcal{V}$, then there are at least two shortest s - t paths. Further, since t is a polygon vertex, due to our general position assumption that any two polygon vertices have no more than one shortest path, $U_s(t)$ has at least two vertices. This implies that s is in a bisector edge of $SPM(t)$. Further, as s is on e_s , s is at the intersection of a bisector edge of $SPM(t)$ and e_s . Correspondingly, we compute the candidate points as follows.

We consider all polygon vertices. For each vertex, we consider it as t and compute $SPM(t)$. For each bisector edge, if it intersects a polygon edge, we compute the intersection as a candidate point. In this way, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time.

2. If $t \in E$, let e_t be the polygon edge that contains t . Since $t \in E$ and t is degenerate, there are at least three shortest paths from s to t . Since $t \in e_t$, $U_t(s)$ must have two vertices v_1 and v_2 that satisfy the condition of Observation 1(2). Hence, $|U_t(s)| \geq 2$. Depending on whether $|U_t(s)| \geq 3$, there are two cases.
 - (a) If $|U_t(s)| \geq 3$, let v_i with $i = 1, 2, 3$ be any three vertices of $U_t(s)$. Then, there are three shortest s - t paths $\pi_{u_i, v_i}(s, t)$ for $i = 1, 2, 3$. We have the following

$$|tv_1| + d(v_1, u_1) + |u_1s| = |tv_2| + d(v_2, u_2) + |u_2s| = |tv_3| + d(v_3, u_3) + |u_3s|.$$

The above equations provide two constraints, and along with the constraints that $s \in e_s$ and $t \in e_t$, the four constraints can determine s and t . Correspondingly, we compute the candidate points as follows.

We enumerate all possible combinations of three vertices as v_i for $i = 1, 2, 3$ and a polygon edge as e_t . For each combination, we compute the overlay of the shortest path maps of v_1 , v_2 , and v_3 . For each cell C of the overlay, if it has an edge that is a portion of a polygon edge, then we consider the polygon edge as e_s and obtain the three roots of the three shortest path maps as u_i for $i = 1, 2, 3$. Next, using the four constraints mentioned above, we can determine a constant number of tuples (\hat{s}, t) and each such \hat{s} is a candidate point. Since the size of the overlay is $O(n^2)$, for each combination we can obtain $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^4)$ combinations, we can compute $O(n^6)$ candidate points in $O(n^6 \log n)$ time.

- (b) If $|U_t(s)| = 2$, let v_1 and v_2 be the two vertices of $U_t(s)$. Since there are at least three shortest s - t paths and due to our general position assumption that no two polygon vertices have more than one shortest path, one of v_1 and v_2 , say v_1 , must have at least two shortest paths to s such that $|U_s(v_1)| \geq 2$. Hence, s is on a bisector edge of $SPM(v_1)$. Since $s \in e_s$, s is the intersection of e_s and a bisector edge of $SPM(v_1)$. Correspondingly, we can compute the candidate points as follows.

We enumerate all polygon vertices. For each vertex, we consider it as v_1 and compute its shortest path map $SPM(v_1)$. Then, for each intersection between a bisector edge of $SPM(v_1)$ and a polygon edge, we consider it as a candidate point. Since the size of $SPM(v_1)$ is $O(n)$, there are $O(n)$ such candidate points. In this way, we can compute at most $O(n^2)$ candidate points in $O(n^2 \log n)$ time.

3. If $t \in \mathcal{I}$, then there are at least four shortest s - t paths, and $|U_t(s)| \geq 3$ by Observation 1. Depending on whether $|U_t(s)| \geq 4$, there two cases.
 - (a) If $|U_t(s)| \geq 4$, then let v_i with $1 \leq i \leq 4$ be any four vertices of $U_t(s)$. Then, there are four shortest s - t paths $\pi_{u_i, v_i}(s, t)$ for $1 \leq i \leq 4$. We have the following

$$\begin{aligned} |tv_1| + d(v_1, u_1) + |u_1s| &= |tv_2| + d(v_2, u_2) + |u_2s| \\ &= |tv_3| + d(v_3, u_3) + |u_3s| = |tv_4| + d(v_4, u_4) + |u_4s|. \end{aligned}$$

The above equations provide three constraints, and along with the constraints that $s \in e_s$, the four constraints can determine s and t . Correspondingly, we can compute $O(n^6)$ candidate points in $O(n^6 \log n)$ time. The algorithm is similar as before and we omit the details.

- (b) If $|U_t(s)| = 3$, let v_i with $i = 1, 2, 3$ be the three vertices of $U_t(s)$. Since there are at least four shortest s - t paths and due to our general position assumption that no two polygon vertices have more than one shortest path, one of v_i with $i = 1, 2, 3$, say v_1 , must have at least two shortest paths to s such that $|U_s(v_1)| \geq 2$. Hence, s is on a bisector edge of $SPM(v_1)$. Since $s \in e_s$, s is the intersection of e_s and a bisector edge of $SPM(v_1)$. Correspondingly, by using the same approach as the above case 2(b), we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time.

This finishes the case for $s \in E$.

This also finishes our algorithms for computing candidate points for the degenerate case. In summary, we can compute a total of $O(n^9)$ candidate points in $O(n^9 \log n)$ time such that s is one of these candidate points.

6 Computing the Geodesic Centers

In this section, we find all geodesic centers from the candidate point set S . Let S' denote the set of candidate points for all cases other than the four dominating cases. Recall that S_d is the set of candidate points for the four dominating cases. Hence, $S = S' \cup S_d$. As discussed in Section 5, $|S'| = O(n^{10})$ and we can find all geodesic centers in S' in $O(n^{11} \log n)$ time by computing their shortest path maps. In the following, we focus on finding all geodesic centers in S_d .

We first remove all points from S_d that are also in S' , which can be done in $O(n^{11} \log n)$ time (e.g., by first sorting these points by their coordinates). Then, for any point $s \in S_d$, if s is a geodesic center, s does not have any degenerate farthest point since otherwise s was also in S' and thus would have already been removed from S_d .

Recall that each point s of S_d is a valid candidate point and we have maintained its path information for s (in particular, the value $d(s)$).

We first perform the following *duplication-cleanup* procedure: for each point $s \in S_d$, if there are many copies of s , we only keep the one with the largest value $d(s)$ (if more than one copy has the largest value, we keep an arbitrary one). This procedure can be done in $O(n^{11} \log n)$ time (e.g., by first sorting all points of S_d by their coordinates). According to our algorithm for computing the candidate points of S_d , we have the following observation.

Observation 7 *After the duplication-cleanup procedure, for any point $s \in S_d$, if s is a geodesic center, then $d_{\max}(s) = d(s)$.*

Proof: For differentiation, we use S_d to denote the original set of S_d before the duplication-cleanup procedure and use S'_d to denote the set after the procedure. Note that S_d and S'_d contain the same physical points and the difference is that for each point of S'_d , S_d may contain multiple copies of the point, and each copy is associated with different path information. Recall that none of the point of S_d is in S' .

Consider any point $s \in S'_d$ and suppose s is a geodesic center. According to our algorithm for computing candidate points of S_d , there must be a copy of s in S_d for which we have maintained its path information that includes a farthest point t of s and $d(s) = d(s, t)$ by Observation 6. Since t is a farthest point of s , $d(s, t) = d_{\max}(s)$. Therefore, there exists a copy of s in S_d with $d(s) = d_{\max}(s)$. Let s' denote any other copy of s in S_d . Again, by Observation 6, $d(s')$ is equal to the shortest path distance from s' to a point t' . Hence, $d(s') \leq d_{\max}(s') = d_{\max}(s) = d(s)$.

According to our duplication-cleanup procedure, if $d(s') < d(s)$, then the copy s' will be removed from S'_d . Otherwise, either copy may be removed, but in either case, the $d(\cdot)$ value of the remaining copy is always equal to $d_{\max}(s)$. Hence, the observation follows. \square

Recall that all points of S_d are in \mathcal{I} .

In the following, we give a *pruning algorithm* that can eliminate most of the points from S_d such that none of these eliminated points is a geodesic center and the number of remaining points of S_d is $O(n^{10})$. Our pruning algorithm relies on the property that each candidate point s of S_d is valid. Specifically, if s is computed for the dominating case $(3, 0, 0)$, then s is associated with the following path information $d(s)$, t_i , v_{ij} , and u_{ij} for $1 \leq i \leq 3$ and $1 \leq j \leq 3$, such that Observation 6 holds (i.e., s is \hat{s} and each t_i is \hat{t}_i). For other three dominating cases (e.g., $(2, 1, 0)$, $(1, 2, 0)$, and $(0, 3, 0)$), there are similar properties. By using these properties, we have the following lemma.

Lemma 17. *Let s be any point in S_d . If s is in the interior of a cell or an edge of \mathcal{D}_{spm} , then for any other point s' in the interior of the same cell or edge of \mathcal{D}_{spm} , it holds that $d_{\max}(s') > d(s)$.*

Proof: The proof uses similar techniques as that for Lemma 3. We only prove the case for s being a candidate point for Case $(3, 0, 0)$, and other cases are similar.

Recall that s is associated with the following path information $d(s)$, t_i , v_{ij} , and u_{ij} for $1 \leq i \leq 3$ and $1 \leq j \leq 3$, such that Observation 6 holds.

If s' be any other point in the same cell or edge of \mathcal{D}_{spm} that contains s . Hence, $SPM(s)$ and $SPM(s')$ are topologically equivalent. Consider any t_i with $1 \leq i \leq 3$. By Observation 6, for each $j = 1, 2, 3$, $\pi_{u_{ij}, v_{ij}}(s, t_i)$ is a shortest paths from s to t_i . Since $SPM(s)$ and $SPM(s')$ are topologically equivalent, $SPM(s')$ has one and only one vertex t'_i corresponding to t_i and for each $j = 1, 2, 3$, $\pi_{u_{ij}, v_{ij}}(s', t'_i)$ is a shortest path from s' to t'_i .

Let r be the direction from s to s' . By Observation 6, $R_\pi(s, t_1) \cap R_\pi(s, t_2) \cap R_\pi(s, t_3) = \emptyset$. Therefore, r is not in $R_\pi(s, t_i)$ for some i with $1 \leq i \leq 3$. Without loss of generality, we assume r is not in $R_\pi(s, t_1)$.

Suppose we move s along the direction r with unit speed and move t_1 to t'_1 with speed $|t_1 t'_1|/|ss'|$. Hence, when s arrives at s' , t_1 arrives at t'_1 simultaneously. Since r is not in $R_\pi(s, t_1)$, there must be a path $\pi_{u_{1j}, v_{1j}}(s, t_1)$ for some j with $1 \leq j \leq 3$ such that $d'_{u_{1j}, v_{1j}}(s, t_1) < 0$ does not hold. Without loss of generality, we assume $j = 1$. In other words, either $d'_{u_{11}, v_{11}}(s, t_1) > 0$ or $d'_{u_{11}, v_{11}}(s, t_1) = 0$.

1. If $d'_{u_{11}, v_{11}}(s, t_1) > 0$, then since $d''_{u_{11}, v_{11}}(s, t_1) \geq 0$ always holds, we have $d_{u_{11}, v_{11}}(s, t_1) < d_{u_{11}, v_{11}}(s', t'_1)$. Since $d(s) = d_{u_{11}, v_{11}}(s, t_1)$ and $d_{\max}(s') \geq d(s', t'_1) = d_{u_{11}, v_{11}}(s', t'_1)$, we obtain that $d_{\max}(s') > d(s)$, which proves the lemma.
2. If $d'_{u_{11}, v_{11}}(s, t_1) = 0$, recall that $d''_{u_{11}, v_{11}}(s, t_1) \geq 0$ always holds. If $d''_{u_{11}, v_{11}}(s, t_1) > 0$, then we again obtain $d_{u_{11}, v_{11}}(s, t_1) < d_{u_{11}, v_{11}}(s', t'_1)$, and consequently, $d_{\max}(s') > d(s)$. Otherwise, $d''_{u_{11}, v_{11}}(s, t_1) = 0$. We claim that this case cannot happen. Indeed, suppose to the contrary that $d''_{u_{11}, v_{11}}(s, t_1) = 0$. Then, we have $d_{u_{11}, v_{11}}(s, t_1) = d_{u_{11}, v_{11}}(s', t'_1)$. Since $\pi_{u_{1j}, v_{1j}}(s', t'_1)$ for $j = 2, 3$ is also a shortest path from s to t'_1 , we have $d_{u_{11}, v_{11}}(s', t'_1) = d_{u_{12}, v_{12}}(s', t'_1) = d_{u_{13}, v_{13}}(s', t'_1)$, and thus, $d_{u_{12}, v_{12}}(s, t_1) = d_{u_{12}, v_{12}}(s', t'_1)$ and $d_{u_{13}, v_{13}}(s, t_1) = d_{u_{13}, v_{13}}(s', t'_1)$. This is only possible if $d'_{u_{1j}, v_{1j}}(s, t_1) = 0$ and $d''_{u_{1j}, v_{1j}}(s, t_1) = 0$ for both $j = 2, 3$. However, by Observation 6, t_1 is in the interior of the triangle $\triangle v_{11}v_{12}v_{13}$; as discussed earlier (i.e., the discussions for Fig. 6), this implies that it is not possible to have $d'_{u_{1j}, v_{1j}}(s, t_1) = 0$ and $d''_{u_{1j}, v_{1j}}(s, t_1) = 0$ for all $j = 1, 2, 3$, incurring contradiction.

This completes the proof of the lemma. \square

Lemma 18. *For any two points s_1 and s_2 of S_d that are in the interior of the same cell or the same edge of \mathcal{D}_{spm} , if $d(s_1) < d(s_2)$, then s_1 cannot be a geodesic center, and if $d(s_1) = d(s_2)$, then neither s_1 nor s_2 is a geodesic center.*

Proof: Consider any two points s_1 and s_2 of S_d that are in the interior of the same cell or the same edge of \mathcal{D}_{spm} . By Lemma 17, $d_{\max}(s_1) > d(s_2)$ and $d_{\max}(s_2) > d(s_1)$.

If $d(s_1) < d(s_2)$, then we obtain $d_{\max}(s_1) > d(s_2) > d(s_1)$. Thus, s_1 cannot be a geodesic center since otherwise $d(s_1)$ would be equal to $d_{\max}(s_1)$ by Observation 7.

Similarly, if $d(s_1) = d(s_2)$, then we have both $d_{\max}(s_1) > d(s_2) = d(s_1)$ and $d_{\max}(s_2) > d(s_1) = d(s_2)$. Thus, neither s_1 nor s_2 is a geodesic center. \square

Based on Lemma 18, our pruning algorithm for S_d works as follows. For each point s of S_d , we determine the cell, edge, or vertex of \mathcal{D}_{spm} that contains s in its interior, which can be done in $O(\log n)$ time by using a point location data structure [9,14] with $O(n^{10})$ time and space preprocessing on \mathcal{D}_{spm} . For each edge or cell, let S'_d be the set of points of S_d that are contained in its interior. We find the point s of S'_d with the largest value $d(s)$. If there are more than one such point in S'_d , we remove all points of S'_d from S_d ; otherwise, remove all points of S'_d except s from S_d . By Lemma 18, none of the points of S_d that are removed above is a geodesic center. After the above pruning algorithm, S_d contains at most one point in the interior of each cell, edge, or vertex of \mathcal{D}_{spm} . Hence, $|S_d| = O(|\mathcal{D}_{spm}|)$. Since $|\mathcal{D}_{spm}| = O(n^{10})$ [7], we obtain $|S_d| = O(n^{10})$.

Theorem 1. *All geodesic centers of \mathcal{P} can be computed in $O(n^{11} \log n)$ time.*

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References

1. H.-K. Ahn, L. Barba, P. Bose, J.-L. De Carufel, M. Korman, and E. Oh. A linear-time algorithm for the geodesic center of a simple polygon. In *Proc. of the 31st Annual Symposium on Computational Geometry (SoCG)*, pages 209–223, 2015.
2. T. Asano and G. Toussaint. Computing the geodesic center of a simple polygon. Technical Report SOCS-85.32, McGill University, Montreal, Canada, 1985.
3. S.W. Bae, M. Korman, and Y. Okamoto. The geodesic diameter of polygonal domains. *Discrete and Computational Geometry*, 50:306–329, 2013.
4. S.W. Bae, M. Korman, and Y. Okamoto. Computing the geodesic centers of a polygonal domain. In *Proc. of the 26th Canadian Conference on Computational Geometry (CCCG)*, 2014. Journal version published online in *Computational Geometry: Theory and Applications*, 2015.
5. S.W. Bae, M. Korman, Y. Okamoto, and H. Wang. Computing the L_1 geodesic diameter and center of a simple polygon in linear time. *Computational Geometry: Theory and Applications*, 48:495–505, 2015.
6. B. Chazelle. A theorem on polygon cutting with applications. In *Proc. of the 23rd Annual Symposium on Foundations of Computer Science*, pages 339–349, 1982.
7. Y.-J. Chiang and J.S.B. Mitchell. Two-point Euclidean shortest path queries in the plane. In *Proc. of the Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 215–224, 1999.
8. H.N. Djidjev, A. Lingas, and J.-R. Sack. An $O(n \log n)$ algorithm for computing the link center of a simple polygon. *Discrete and Computational Geometry*, 8:131–152, 1992.
9. H. Edelsbrunner, L. Guibas, and J. Stolfi. Optimal point location in a monotone subdivision. *SIAM Journal on Computing*, 15(2):317–340, 1986.
10. D. Halperin and M. Sharir. New bounds for lower envelopes in three dimensions, with applications to visibility in terrains. *Discrete and Computational Geometry*, 12:313–326, 1994.
11. J. Hershberger and S. Suri. Matrix searching with the shortest-path metric. *SIAM Journal on Computing*, 26(6):1612–1634, 1997.
12. J. Hershberger and S. Suri. An optimal algorithm for Euclidean shortest paths in the plane. *SIAM Journal on Computing*, 28(6):2215–2256, 1999.
13. Y. Ke. An efficient algorithm for link-distance problems. In *Proc. of the 5th Annual Symposium on Computational Geometry*, pages 69–78, 1989.
14. D. Kirkpatrick. Optimal search in planar subdivisions. *SIAM Journal on Computing*, 12(1):28–35, 1983.
15. J.S.B. Mitchell. Geometric shortest paths and network optimization, in *Handbook of Computational Geometry*, J.-R. Sack and J. Urrutia (eds.), pages 633–702. Elsevier, Amsterdam, the Netherlands, 2000.
16. B.J. Nilsson and S. Schuierer. Computing the rectilinear link diameter of a polygon. In *Proc. of the International Workshop on Computational Geometry - Methods, Algorithms and Applications*, pages 203–215, 1991.
17. B.J. Nilsson and S. Schuierer. An optimal algorithm for the rectilinear link center of a rectilinear polygon. *Computational Geometry*, 6:169–194, 1996.
18. R. Pollack, M. Sharir, and G. Rote. Computing the geodesic center of a simple polygon. *Discrete and Computational Geometry*, 4(1):611–626, 1989.

19. S. Schuierer. An optimal data structure for shortest rectilinear path queries in a simple rectilinear polygon. *International Journal of Computational Geometry and Applications*, 6:205–226, 1996.
20. S. Suri. Computing geodesic furthest neighbors in simple polygons. *Journal of Computer and System Sciences*, 39:220–235, 1989.